Master thesis

p-adic integration theories

Stefan Stadlöder

19 October 2017

Prof. Dr. Walter Gubler

Faculty of Mathematics

University of Regensburg

Contents

Zusammenfassung

Ziel dieser Arbeit ist, das p-adische Analogon des komplexen Wegintegrals zu untersuchen. Dabei werden zwei verschiedene Ansätze, das abelsche Integral und das Berkovich-Coleman-Integral, eingeführt und anschließend verglichen. Weil das abelsche Integral jedoch nur auf Kurven definiert ist, wird sich in dieser Arbeit auch auf diese beschränkt.
p-adische Zahlen haben im Gegensatz zu den reellen oder komplexen Zahlen die besondere Eigenschaft, dass sie eine total unzusammenhängende Topologie besitzen, das heißt die einzigen zusammenhängenden Teilmengen sind die leere Menge und Mengen, die nur aus einem Element bestehen. Das macht es so schwer, einen Weg auf einer p-adischen Kurve zu definieren, da die bisherige Definition eines Wegs im archimedischen Fall nicht angewendet werden kann. Weiterhin ergibt sich auf einer p-adischen Kurve, die in den eindimensionalen affinen Raum eingebettet werden kann, das Problem, dass zwei offene Bälle entweder disjunkt sind oder einer den anderen enthält. Das heißt der Weg von einem Punkt P zu einem anderen Punkt Q kann entweder durch einen einzelnen offenen Ball überdeckt werden oder es ist gänzlich unmöglich, ihn mit offenen Bällen zu überdecken. Offene Bälle haben die besondere Eigenschaft, dass auf ihnen das Poincaré-Lemma angewendet werden kann. Dieses besagt, dass für eine geschlossene Differentialform ω eine Stammfunktion F existiert. Im ersten Fall kann man das Integral also ganz einfach als

$$\int_P^Q \omega = \int_P^Q \mathrm{d}F = F(Q) - F(P)$$

berechnen, wofür man noch nicht einmal einen Weg von P nach Q braucht. Das gilt auch tatsächlich für alle p-adischen Integrationstheorien. Im zweiten Fall kann man keine Stammfunktionen von ω auf dem ganzen Weg von P nach Q bilden, sofern man es überhaupt schafft, einen Weg zu definieren.
Man beginnt zunächst allgemein zu definieren, was eine p-adische Integrationstheorie sein soll, indem man die wichtigsten Eigenschaften des komplexen Wegintegrals nimmt und diese auch für den p-adischen Fall fordert. Nun könnten rein theoretisch unendlich viele verschiedene p-adische Integrationstheorien existieren, die diese Anforderungen erfüllen. Bis heute existieren davon allerdings erst zwei: das abelsche Integral und das Berkovich-Coleman-Integral.
Ersteres löst die beiden genannten Probleme, indem es gänzlich auf Wege verzichtet und, statt mit der Kurve selbst, mit seiner Jacobischen J arbeitet. Man bildet also die $\mathbb{C}_p$-rationalen Punkte P und Q mit Hilfe der Abel-Jacobi-Abbildung ι bezüglich eines beliebigen Basispunkts auf J ab. Die Jacobische hat nun die besondere Eigenschaft, dass sie eine abelsche Varietät ist. Auf deren $\mathbb{C}_p$-rationalen Punkten lässt sich nun ein

universeller Logarithmus

$$\log_J : J(\mathbb{C}_p) \longrightarrow \mathrm{Lie}(J)$$

definieren. Dieser ist auf ganz $J(\mathbb{C}_p)$ definiert und eindeutig. Er lässt sich aber nur mit Hilfe p-adischer Lie-Theorie definieren, welche sich anwenden lässt, weil $J(\mathbb{C}_p)$ eine p-adische Lie-Gruppe ist. Eine Lie-Gruppe ist eine Mannigfaltigkeit mit einer Gruppenstruktur. $\mathrm{Lie}(J)$ ist die zu $J(\mathbb{C}_p)$ gehörige Lie-Algebra, deren zugrundeliegende Menge der Tangentialraum von $J(\mathbb{C}_p)$ in 0 ist. Diese kann schließlich mit der zu $\Omega^1_{J/\mathbb{C}_p}$, dem Raum der Differentialformen vom Grad 1, dualen Gruppe identifiziert werden. Das heißt es ist möglich, $\log_J(P)$ und $\log_J(Q)$ als Homomorphismen von $\Omega^1_{J/\mathbb{C}_p}$ nach $\mathbb{C}_p$ zu betrachten. Damit wird das abelsche Integral auf $J(\mathbb{C}_p)$ als

$$^{\mathrm{Ab}}\int_P^Q \omega = (\log_J(Q))(\omega) - (\log_J(P))(\omega)$$

definiert. Mittels Rücktransport kann das Integral schließlich auf die Kurve zurückgezogen werden, da ι^* ein Isomorphismus zwischen dem Raum der Differentialformen vom Grad 1 von J und der Kurve ist. Da alle Differentialformen auf J zudem translationsinvariant sind, ist diese Definition sogar unabhängig von der Wahl des Basispunkts von ι.

Der zweite Ansatz, das Berkovich-Coleman-Integral, verfolgt eine ganz andere Herangehensweise. So wird hier nicht die Kurve X selbst, sondern seine Analytifizierung X^{an} betrachtet. Dies macht es möglich, Wege zwischen $\mathbb{C}_p$-rationalen Punkten P und Q zu definieren, da die Analytifizierung eine Verbindung zwischen den einzelnen Zusammenhangskomponenten, die alle nur aus einem Punkt bestehen, herstellt. Den Grundstein dieser Theorie legte Robert F. Coleman in den 1980er Jahren, der damals noch die Sprache der rigiden Analysis benutzte. Erst später wurde diese durch die Theorie der Berkovich-Räume abgelöst. Vladimir G. Berkovich war es auch, der die Theorie von Coleman für $\mathbb{C}_p$-analytische Räume verallgemeinerte, sodass man nicht mehr nur auf Kurven beschränkt ist. In dieser Arbeit werden jedoch auch die Resultate von Coleman in der Sprache von Berkovich formuliert.

Man kann mit Hilfe eines sogenannten Modells $\mathcal{X}$ die Analytifizierung der Kurve X in drei verschiedene Arten von disjunkten Teilmengen unterteilen, deren Vereinigung wiederum X^{an} ergibt. Diese sind offene Bälle, offene Kreisringe oder Knoten, das heißt einelementige Mengen. Die einelementigen Mengen bestehen aus nichtrationalen Punkten, die als Endpunkte nicht in Frage kommen. Auf offenen Bällen ist die Integration dank des Poincaré-Lemmas unproblematisch. Auf offenen Kreisringen sind die Differentialformen vom Grad 1 durch konvergente Laurentreihen aus $\mathbb{C}_p[[T, T^{-1}]]$ gegeben. Diese kann man auch integrieren, jedoch nur, falls der Koeffizient von $\frac{1}{T}\mathrm{d}T$ gleich 0 ist. Zuerst wurde dieses Problem gelöst, indem man einen p-adischen Logarithmus konstruierte, der eine Stammfunktion von $\frac{1}{T}\mathrm{d}T$ darstellte. Dieser hat jedoch, genau wie der komplexe Logarithmus, unendlich viele Zweige. Ein Zweig ist durch den Wert von $\mathrm{Log}(p) \in \mathbb{C}_p$ definiert. Coleman fixierte hierfür einen Wert und beließ diesen während des gesamten

Integrationsprozesses gleich. Die Wahl des Zweiges ist nicht entscheidend, es ist nur wichtig, dass man diesen nicht ändert. Damit kann man auch auf den offenen Kreisringen integrieren.
Man will jedoch auch zwischen Punkten aus verschiedenen offenen Bällen oder Kreisringen integrieren können. Dazu muss man die Knoten passieren, die Bindeglieder zwischen den offenen Bällen und Kreisringen darstellen. Dieses Problem konnte von Coleman gelöst werden. Man betrachtet einen einzelnen Knoten und definiert sich diesen und alle angrenzenden offenen Bälle und Kreisringe als ein weites offenes Teilgebiet dieses Knotens, welches einfach-zusammenhängend ist, das heißt alle Wege zwischen P und Q sind homotopieäquivalent. Man kann nun für P beziehungsweise Q eine Stammfunktion F_1 beziehungsweise F_2 auf dem zugehörigen offenen Ball oder Kreisring finden. Das Problem dabei ist, dass die Definition

$$^{\mathrm{BC}}\int_P^Q \omega = F_2(Q) - F_1(P)$$

nicht wohldefiniert ist, da die Stammfunktionen nur bis auf eine Konstante definiert sind, die sich hier nicht aufhebt, da es sich um zwei verschiedene Stammfunktionen handelt. Coleman schaffte es schließlich, diese Konstante mit Hilfe von logarithmischen F-Kristallen in den Griff zu bekommen, sodass nun mit obiger Definition sinnvoll gearbeitet werden kann.
Die Arbeit von Coleman wurde später von Vladimir G. Berkovich auf $\mathbb{C}_p$-analytische Räume verallgemeinert. Aufgrund der Komplexität dieser Theorie beschränkt sich diese Arbeit aber darauf, die wichtigsten Unterschiede und Gemeinsamkeiten zu Colemans Definition aufzuzeigen.
Im letzten Kapitel wird schließlich untersucht, ob und wann beide Integrale übereinstimmen. Als Grundlage hierfür dient ein Theorem, das besagt, dass sich die Differenz beider Integrale als Verknüpfung einer linearen Abbildung L und der Tropikalisierungsabbildung trop darstellen lässt. Als unmittelbare Konsequenz daraus ergibt sich die Übereinstimmung beider Integrale auf abelschen Varietäten guter Reduktion sowie auf offenen Bällen in X^{an}. Das Hauptresultat dieser Arbeit beschreibt schließlich den Vergleich auf offenen Kreisringen. Es besagt, dass eine $\mathbb{C}_p$-lineare Abbildung

$$a : \Omega^1_{X/\mathbb{C}_p} \longrightarrow \mathbb{C}_p$$

existiert, sodass sich für die Differenz der Ausdruck

$$^{\mathrm{BC}}\int_P^Q \omega - {}^{\mathrm{Ab}}\int_P^Q \omega = a(\omega)\,(v(Q) - v(P))\,.$$

ergibt, wobei v die p-adische Bewertung ist. Diese kann im offenen Kreisring angewendet werden, da hier P und Q isomorph zu p-adischen Zahlen sind. Für $n := \dim_{\mathbb{C}_p}(\Omega^1_{X/\mathbb{C}_p}) > 1$ ergibt sich damit ein mindestens $(n-1)$-dimensionaler Unterraum von $\Omega^1_{X/\mathbb{C}_p}$, auf dem beide Integrale übereinstimmen. Daraus ergeben sich wichtige Resultate für die Beschränkung der $\mathbb{Q}$-rationalen Punkte auf Kurven von kleinem Mordell-Weil-Rang (vgl. Theorem 9.1, [Sto13] und Theorem 2.14, [KRZ16a]).

1 Introduction

1.1 The history of p-adic numbers

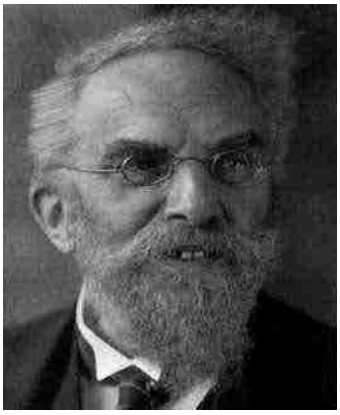


Figure 1.1: Kurt Hensel (1861-1941)
Source: http://www.learn-math.info/historyDetail.htm?id=Hensel (visited on Oct. 18, 2017)

p-adic numbers were developed by Kurt Hensel towards the end of the 19^{th} century. He started defining the p-adic absolute value

$$|.|_p : \mathbb{Q} \longrightarrow \mathbb{Q}$$

with respect to a prime number $p \in \mathbb{N}$ in the following way. Every rational number $q \neq 0$ has a unique representation

$$q = \pm \tfrac{a}{b} p^n$$

with $n \in \mathbb{Z}$, $a, b \in \mathbb{N}\backslash\{0\}$, $p \nmid a, b$ and a, b coprime. Then he defined

$$|q|_p := p^{-n} \text{ and } |0|_p = 0.$$

For $3.6 = \frac{18}{5} = \frac{2 \cdot 3^2}{5}$, one gets $\left|\frac{18}{5}\right|_2 = \frac{1}{2}$, $\left|\frac{18}{5}\right|_3 = \frac{1}{9}$, $\left|\frac{18}{5}\right|_5 = 5$ and $\left|\frac{18}{5}\right|_p = 1$ for all other p prime. This is a non-archimedean absolute value, which means that the ultrametric triangle inequality

$$|x + y|_p \leq \max(|x|_p, |y|_p)$$

holds for $x, y \in \mathbb{Q}$. In contrast to this, the more popular absolute value $|.|_\infty$, which just eliminates the minus-sign, is an archimedean absolute value, as

$$|1+2|_\infty = 3 > \max(|1|_\infty, |2|_\infty)$$

and thus the ultrametric triangle inequality does not hold.
Ostrowski's theorem says that $|.|_\infty$ and $|.|_p$ for p prime are, up to equivalence, the only absolute values on $\mathbb{Q}$. Since every absolute value on $\mathbb{Q}$ yields a metric on $\mathbb{Q}$, one can complete $\mathbb{Q}$ with respect to the absolute values $|.|_\infty$ and $|.|_p$. It is established that the completion of $\mathbb{Q}$ with respect to $|.|_\infty$ is $\mathbb{R}$, the set of real numbers, where new numbers, such as $\sqrt{2} \notin \mathbb{Q}$, arise. Doing the completion of $\mathbb{Q}$ with respect to $|.|_p$, one obtains a new field, denoted by $\mathbb{Q}_p$. The numbers that arise by this completion can be represented by convergent power series of the form

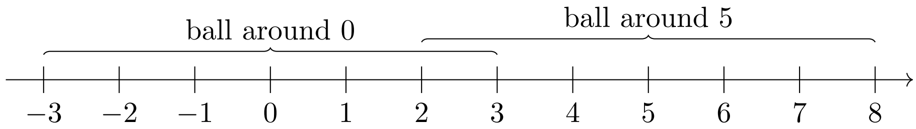
$$\sum_{i=i_0}^{\infty} a_i p^i$$

with $i_0 \in \mathbb{Z}$ and $a_i \in \{1, ..., p-1\}$. If it holds $a_i = 0$ for all $i > n$ for a certain $n \in \mathbb{Z}$, then the element is in $\mathbb{Q}$. With respect to the archimedean absolute value $|.|_\infty$, these new numbers would be "∞" because the power series would not converge in $\mathbb{R}$.
If one takes the algebraic closure of $\mathbb{R}$ respectively $\mathbb{Q}_p$ and completes this afterwards, one obtains the fields $\mathbb{C}$ respectively $\mathbb{C}_p$.

$$\begin{tikzcd}
 & & \mathbb{Q} \arrow[dl, "|\cdot|_\infty"'] \arrow[dr, "|\cdot|_p"] & \\
\text{completion} & \mathbb{R} \arrow[d, "|\cdot|_\infty"] & & \mathbb{Q}_p \arrow[d, "|\cdot|_p"] \\
\text{algebraic closure} & \mathbb{C} \arrow[d, "|\cdot|_\infty"] & & \overline{\mathbb{Q}}_p \arrow[d, "|\cdot|_p"] \\
\text{completion} & \mathbb{C} & & \mathbb{C}_p
\end{tikzcd}$$

Therefore $\mathbb{C}_p$ may be considered as the p-adic analogue of the complex numbers.
The p-adic numbers $\mathbb{Q}_p$ became a powerful tool in solving diophantic equations, which have already been a point of interest in number theory for many centuries. By the local-global principle of Hasse and Minkowski for example, a quadratic form over a number field can be solved in $\mathbb{Q}$ if and only if it can be solved in $\mathbb{R}$ and all $\mathbb{Q}_p$. The importance of p-adic numbers may be underlined by the fact that totally new areas of research in non-archimedean geometry have arisen since then, for instance Berkovich theory in the 1980s.
Investigating the topology of p-adic numbers, a couple of odd properties arise: On the

following number line $\mathbb{R}$, two open balls of radius 3 around 0 and 5 are drawn:



The intersection of both balls is exactly the open interval $(2, 3)$. This holds because for the numbers $x \in (2, 3)$ we have $|0 - x|_\infty < 3$ and $|5 - x|_\infty < 3$.
With respect to the p-adic absolute value $|.|_p$, this phenonemon is not possible anymore. If one takes $p = 5$ and the numbers 0 and $\frac{1}{5}$, then the two numbers have distance $|\frac{1}{5} - 0|_5 = 5$. Considering now two open balls with radius $2.5 < r \leq 5$ around them, they do not intersect anymore. On the assumption that such an element $x \in \mathbb{Q}$ exists, it would have to fulfil $|x - 0|_5 < 5$ and $|x - \frac{1}{5}|_5 < 5$. This contradicts the ultrametric triangle inequality

$$5 = |\tfrac{1}{5} - 0|_5 \leq \max(|\tfrac{1}{5} - x|_5, |x - 0|_5) < 5.$$

Taking now $r > 5$, a new phenomenon arises. An element that lies in one of the open balls, meaning $|x - 0|_5 < r$, lies also in the other open ball, since the ultrametric triangle inequality delivers

$$|x - \tfrac{1}{5}|_5 \leq \max(|x - 0|_5, |0 - \tfrac{1}{5}|_5) < r.$$

Hence, both balls coincide. If the two open balls had different radii $r_1 > r_2 > 5$, the bigger ball would contain the smaller one. That means that in the p-adic topology of the affine line, two open balls are either disjoint or one ball contains the other one.

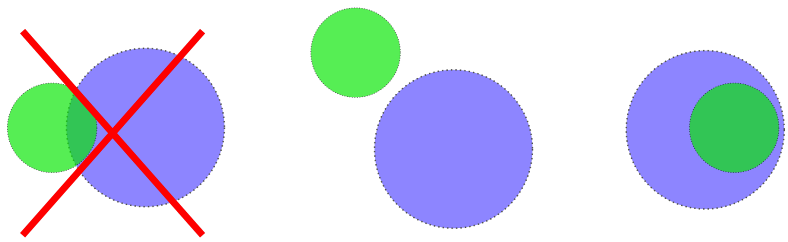


Figure 1.2: The left situation is not possible in p-adic topology

1.2 Path integral on a curve over p-adic numbers

This phenomenon leads to a big problem when trying to construct a path integral on a curve in the p-adic world. The goal is to have an analogue to the complex path integral. Given a smooth, proper, connected $\mathbb{C}_p$-curve X, the aim is to have a map

$$\begin{aligned} \int : \mathcal{P}(X) \times \Omega^1_{X/\mathbb{C}_p} &\longrightarrow \mathbb{C}_p \\ (\gamma, \omega) &\longmapsto \int_\gamma \omega \end{aligned}$$

where $\mathcal{P}(X)$ is the set of all paths in X with $\mathbb{C}_p$-rational end points and $\Omega^1_{X/\mathbb{C}_p}$ the set of all differential one-forms on X.
The first problem is the definition of $\mathcal{P}(X)$. Normally a path γ is a continuous map from a closed interval to X. But $\mathbb{Q}_p$ is not ordered; hence defining a closed interval $[a,b]$, with $a,b \in \mathbb{Q}_p$, does not make sense. Another way would be taking the closed interval in $\mathbb{R}$. But the p-adic topology is totally disconnected, which means that the only connected subsets of X are the singletons and the empty set. Since the image of a connected subset such as $[a,b]$, with $a,b \in \mathbb{R}$, under a continuous map is again connected, the images of such paths can only be singletons, which just correspond to the trivial cases and gives only zero integrals.
The second problem is the application of the Poincaré lemma. It says in particular that, on an open ball, any closed differential one form ($\mathrm{d}\omega = 0$) is exact, meaning that there exists an analytic function f, such that $\omega = \mathrm{d}f$. Given a differential one-form $\omega \in \Omega^1_{X/\mathbb{C}_p}$, one can, in the archimedean case, cover the image of the path by open balls around every point of $\gamma([a,b])$. Since $[a,b]$ is compact and γ is continuous, $\gamma([a,b])$ is also compact. Therefore one may choose just a finite number of these open balls, such that they still cover the path.

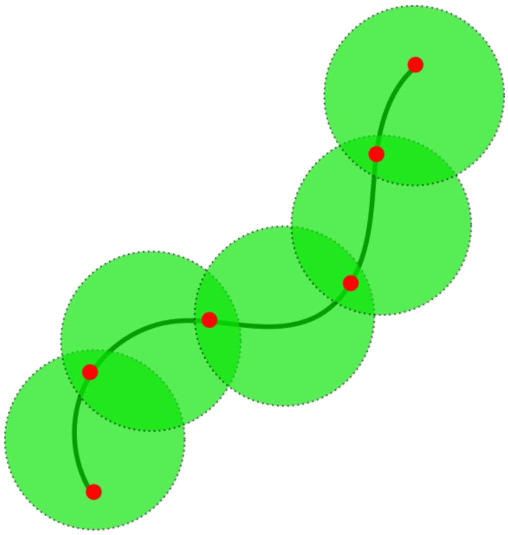


Figure 1.3: Covering the path with finitely many open balls

One may now choose points in the intersections, divide the path by these points, and calculate the integral for the divided paths on the open balls with the Poincaré lemma, which provides an analytic function f with $\omega = \mathrm{d}f$, and hence

$$\int_P^Q \omega = \int_P^Q \mathrm{d}f = f(Q) - f(P).$$

In p-adic topology this is not possible in general. For instance in the one-dimensional affine space $\mathbb{A}^1_{\mathbb{C}_p}$ two open balls are either disjoint or one open ball contains the other one. This means that the path can either be covered by a single open ball, or it is not possible to cover it by open balls at all. The first case would be very easy but the second case leads to a very big problem. If the curve cannot be embedded in $\mathbb{A}^1_{\mathbb{C}_p}$, there may be a possibility to cover the path by open balls, but in contrast to the archimedean case this property does not hold in general. Therefore the goal is to find a way to evade these problems.

One approach doing this was the abelian integral, which was treated by Yuri G. Zarhin in great generality in his paper *p-adic abelian integrals and commutative Lie Groups*, [Zar96], and was generalized by Pierre Colmez. It is possible to avoid these problems by passing to the Jacobian of the curve, which is an abelian variety, and define the integral for abelian varieties. On an abelian variety A, p-adic Lie theory was used in order to define a logarithm on $A(\mathbb{C}_p)$, which is a p-adic Lie group. Normally the logarithm is just defined on a small open subset of $A(\mathbb{C}_p)$, denoted by $A(\mathbb{C}_p)_f$. These are all points $x \in A(\mathbb{C}_p)$, for which there exists a sequence $(n_i)_{i\in\mathbb{N}}$ of non-zero natural numbers, such that

$$n_i x \xrightarrow[i\to+\infty]{} 0.$$

For these points it is now possible to uniquely define a logarithm because the logarithm map in a small neighbourhood U of 0 is defined as the inverse of the exponential mapping. In contrast to the archimedean case, here an exponential mapping is not always unique, but very surprisingly the logarithm defined in this way is unique in U. For points $x \in A(\mathbb{C}_p)_f$ it is now possible to define $\log(x)$ by taking the above limit to 0, where the logarithm is well-defined. This gives an abelian logarithm

$$\log_{\mathrm{Ab}} : A(\mathbb{C}_p)_f \longrightarrow \mathrm{Lie}(A)$$

to the $\mathbb{C}_p$-Lie algebra of $A(\mathbb{C}_p)$. Abelian varieties have the nice property that for them

$$A(\mathbb{C}_p) = A(\mathbb{C}_p)_f$$

always holds, which yields a unique and universally well-defined abelian logarithm

$$\log_{\mathrm{Ab}} : A(\mathbb{C}_p) \longrightarrow \mathrm{Lie}(A).$$

Finally, every image $\log(P)$ of $\mathbb{C}_p$-rational points under this logarithm is a homomorphism from $\Omega^1_{A/\mathbb{C}_p}$ to $\mathbb{C}_p$. The abelian integral is then defined as

$$^{\mathrm{Ab}}\int_P^Q \omega = \log(Q)(\omega) - \log(P)(\omega)$$

and one can produce the integral on the curve by pullback. This version totally ignores the paths and just uses the end points.

Another approach was started by Robert F. Coleman in the 1980s. He used the Frobenius homomorphism to continue the integration from open balls to the whole curve. For this purpose he used the language of rigid analysis, which, meanwhile, is already out of date.

A more modern language was developed by Vladimir G. Berkovich and published in his book *Spectral Theory and Analytic Geometry over Non-Archimedean Fields* [Ber90] in 1990. Berkovich started by taking the idea of Coleman, translating it into Berkovich language and generalizing it. Since Berkovich theory is more modern, the ideas of Coleman will also be described in this language.
Coleman tackled the problem that, on the curve X, it is not possible to define paths $\mathcal{P}(X)$ with $\mathbb{C}_p$-rational end points. The p-adic topology is totally disconnected, meaning that one may imagine the p-adic numbers as a cloud of single points without order. The same holds for the curve X. But with Berkovich theory there is a way to connect these points. For this purpose one has to identify every element $a \in \mathbb{C}_p$ with a map

$$\begin{aligned} \zeta_{a,0} : \mathbb{C}_p[x] &\longrightarrow \mathbb{C}_p, \\ f(x) &\longmapsto f(a). \end{aligned}$$

Furthermore the following maps were defined

$$\begin{aligned} \zeta_{a,r} : \mathbb{C}_p[x] &\longrightarrow \mathbb{C}_p, \\ f(x) &\longmapsto \sup_{|y-a|_p \leq r} |f(y)|_p \end{aligned}$$

for $r \in \mathbb{R}_{>0}$, which are multiplicative seminorms on $\mathbb{C}_p[x]$, continuing the p-adic absolute value. Taking two numbers, for example 0 and $\frac{1}{5}$ from the example in section 1.1, the open balls around them are disjoint for radius $r \leq \left|\frac{1}{5} - 0\right|_5 = 5$ and coincide for $r > 5$. For closed balls they already coincide for $r = 5$. Hence $\zeta_{0,5} = \zeta_{\frac{1}{5},5}$ and $\zeta_{0,r} \neq \zeta_{\frac{1}{5},r}$ for $r < 5$. Extending $\mathbb{C}_5$ by the elements $\zeta_{a,r}$, it is possible to find paths

$$\begin{aligned} \gamma_1 : [0,5] &\longrightarrow \{\text{multiplicative seminorm on } \mathbb{C}_5[x] \text{ continuing } |.|_5\} \\ r &\longmapsto \zeta_{0,r} \end{aligned}$$

and

$$\begin{aligned} \gamma_2 : [0,5] &\longrightarrow \{\text{multiplicative seminorm on } \mathbb{C}_5[x] \text{ continuing } |.|_5\} \\ r &\longmapsto \zeta_{\frac{1}{5},r} \end{aligned}$$

from 0 respectively $\frac{1}{5}$ to $\zeta_{0,5} = \zeta_{\frac{1}{5},5}$. The map

$$\begin{aligned} \gamma : [0,10] &\longrightarrow \{\text{multiplicative seminorm on } \mathbb{C}_5[x] \text{ continuing } |.|_5\} \\ r &\longmapsto \begin{cases} \zeta_{0,r} & \text{for } r \leq 5 \\ \zeta_{\frac{1}{5},10-r} & \text{for } r > 5. \end{cases} \end{aligned}$$

finally builds a path from 0 to $\frac{1}{5}$.

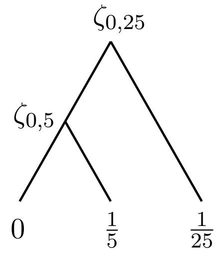


Figure 1.4: Connecting 0, $\frac{1}{5}$ and $\frac{1}{25}$ via elements $\zeta_{a,r}$

The distance between $\frac{1}{25}$ and 0 is $\left|\frac{1}{25} - 0\right|_5 = 25$. Adding this to the picture, it would mean that $\left|\frac{1}{25} - \frac{1}{5}\right|_5 = 25$, too. And this is indeed true as, with the ultrametric triangle inequality, it follows that

$$\left|\frac{1}{25} - \frac{1}{5}\right|_5 \leq \max\left(\left|\frac{1}{25} - 0\right|_5, \left|0 - \frac{1}{5}\right|_5\right) = \max(25, 5) = 25.$$

Since $25 \neq 5$, equality is achieved in the ultrametric triangle inequality, and the picture still makes sense. By this process, called analytification, one may transform the open unit ball $\{x \in \mathbb{C}_p \mid |x|_p < 1\}$ or the open annulus $\{x \in \mathbb{C}_p \mid \rho < |x|_p < 1\}$ for $\rho \in \mathbb{Q}$ and $0 < \rho < 1$ into their analytification $B(1)_+$ and $S(\rho)_+$, which look like trees.

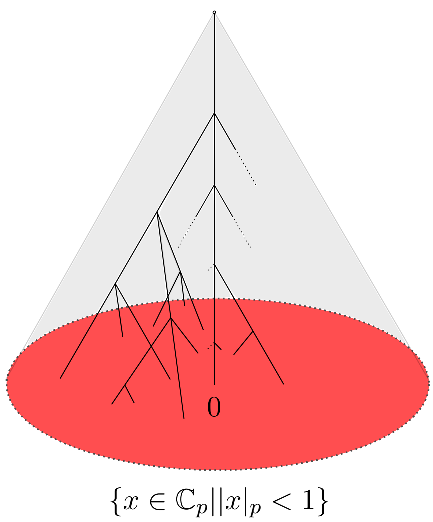


Figure 1.5: Sketch of $B(1)_+$; the $\mathbb{C}_p$-rational points are marked in red

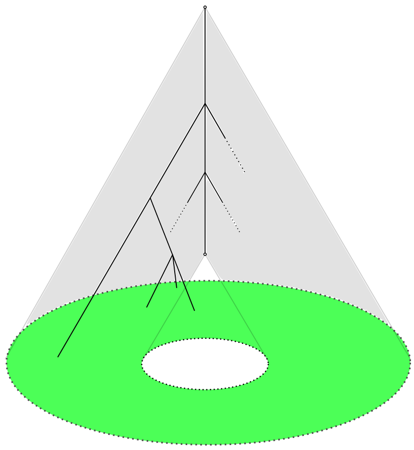


Figure 1.6: Sketch of $S(\rho)_+$; the $\mathbb{C}_p$-rational points are marked in green

Only the trees in black build the analytifications. These are now path-connected. In Figure 1.5, the line from the top to the point 0 may be associated with length 1 as this is the radius of the open ball. In Figure 1.6, this line gets cut as the open ball with radius ρ was removed. The remaining length is only $1 - \rho > 0$. This line will play an important role later as its end points are missing.

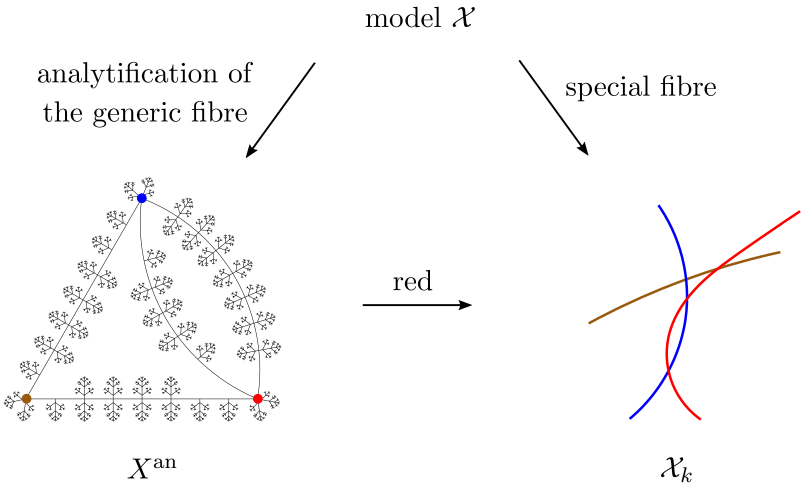


Figure 1.7: Interplay of the model $\mathcal{X}$ and the curve X

A proper, connected, flat R-scheme $\mathcal{X}$ of relative dimension 1 is called a **semistable model** of X if its generic fibre is equal to the smooth curve X and the only singularities of its special fibre $\mathcal{X}_k$ are double points. By a result of Berkovich and Bosch-Lütkebohmert (Theorem 3.2.4, [KRZ16a]), there exists a reduction map from the analytification X^{an} of the curve X to $\mathcal{X}_k$ such that the pre-image of a generic point is a singleton, the pre-image of a smooth point is $B(1)_+$, and the pre-image of a double point is $S(\rho)_+$.

The $\mathbb{C}_p$-rational points of X correspond to the end points of the trees in the analytification X^{an} where it is possible to consider paths as it is path-connected. The coloured vertices are the pre-images of the generic points of the corresponding irreducible components.

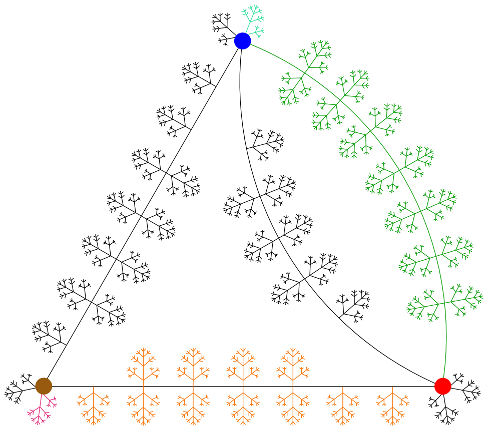


Figure 1.8: Analytification of X

For example, the green set is the pre-image of an intersection point of the blue and red irreducible component; hence it is located between the blue and red vertices. Furthermore it is isomorphic to the open annulus $S(\rho)_+$ for a certain $\rho > 0$. The same holds for the orange set, which comes from an intersection of the brown and red component. The turquoise respectively pink set is the pre-image of a smooth point on the blue respectively brown component and is isomorphic to $B(1)_+$. This is the setting in which will be worked.

After solving the problem with paths, Coleman also tackled the second one. So far it was possible only to integrate within an open ball, for instance the pink set. By defining a logarithm on an open annulus around 0, and fixing the branch of the logarithm, one can also integrate the differential one-form $\frac{\mathrm{d}z}{z}$ and therefore Laurent series on the open annuli. But, even with this, it is still not possible to integrate between different open balls or annuli.

For this reason, Coleman extended the integration to basic wide open subdomains. A basic wide open subdomain is the union of a vertex and all the open balls and open annuli adjacent to it. Let P and Q be two $\mathbb{C}_p$-rational points lying in two different open balls or annuli in a basic wide open subdomain. For $\omega \in \Omega^1_{X/\mathbb{C}_p}$ it holds $\omega = \mathrm{d}F_1$ respectively $\omega = \mathrm{d}F_2$ on the open ball or annulus of P respectively Q because on open balls and annuli it has been shown that antiderivatives always exist. Before the idea of Coleman, the integral

$$^{\mathrm{BC}}\!\!\int_P^Q \omega = F_2(Q) - F_1(P)$$

was not well-defined, as both F_1 and F_2 are just defined up to a constant and, therefore, their difference is not fixed.
But Coleman was able to fix the difference between F_1 and F_2 in the following way: By using the Frobenius homomorphism, he constructed for any basic wide open subdomain a so-called logarithmic F-crystal, which is a module of naive analytic functions. Naive analytic functions are the Berkovich analogue of locally analytic functions. Any differential one-form restricted to a basic wide open subdomain has then an antiderivative in the logarithmic F-crystal belonging to this basic wide open subdomain. This antiderivative is finally used for calculating the integral on a basic wide open subdomain.

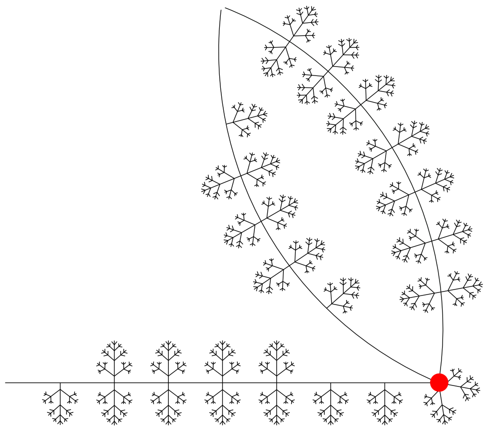


Figure 1.9: Basic wide open subdomain of the red vertex

As a final step, one can cover X^{an} by basic wide open subdomains and gets an integration theory between two $\mathbb{C}_p$-rational points, called the Berkovich-Coleman integral and denoted by ${}^{\text{BC}}\!\int_\gamma \omega$.
Having two integration theories, it is now of interest whether they coincide or deliver in some cases different results. At first, on open balls the integration theories coincide by the Poincaré lemma. But on the whole X^{an} it was not possible to compare them, until Michael Stoll proved in his paper *Uniform bounds for the number of rational points on hyperelliptic curves of small Mordell-Weil rank* (Proposition 7.3, [Sto13]) the following theorem:

Theorem 6.3.6. Let A be an open annulus in X^{an}. Then there exists a $\mathbb{C}_p$-linear map

$$a : \Omega^1_{X/\mathbb{C}_p} \longrightarrow \mathbb{C}_p$$

such that, for all $P, Q \in A(\mathbb{C}_p)$, it holds

$${}^{\text{BC}}\!\int_P^Q \omega - {}^{\text{Ab}}\!\int_P^Q \omega = a(\omega)\,(v(Q) - v(P))\,.$$

Firstly it is unclear what $v(P)$ respectively $v(Q)$ means. In Figure 1.6 the vertical line in the middle had length $1-\rho$. The map

$$\zeta_{0,r} \longmapsto r$$

associates it to the interval $(\rho, 1)$. For the cross points $\zeta_{a,r}$, it always holds $r = p^q$ with $q \in \mathbb{Q}$. Every $\mathbb{C}_p$-rational point may be mapped to a unique point on this line by going to the top along the branches in the tree. After this, the line may be mapped with the negative of the logarithm, that is an isomorphism on $(\rho, 1)$, to the interval $(0, -\log_p(\rho))$. Finally, the image under this isomorphism delivers the value of $v(P)$ respectively $v(Q)$, that is in $\mathbb{Q}$ since $-\log_p(r) = -\log_p(p^q) = q$ for every cross point $\zeta_{a,r}$.
Another question is: Why it is possible to write ${}^{\mathrm{BC}}\!\int_P^Q \omega$ although one needs a path $\gamma \in \mathcal{P}(X)$? Restriction to one open annulus, which is simply-connected as there are no loops, makes the integral path-independent because all paths are homotopy equivalent. The result says now that the difference of the integrals, if they do not coincide, is bigger if for instance in the green open annulus the point P lies near the blue vertex and Q lies near the red one. On top of that, the difference can be changed if, for example, one multiplies ω with a constant. It helps to control the difference between the integrals. Especially $\dim_{\mathbb{C}_p}(\Omega^1_{X/\mathbb{C}_p}) > 1$ delivers a whole subspace of differential one-forms for which the integrals coincide. Why is this helpful?
The abelian and Berkovich-Coleman integral have various advantages and disadvantages, summed up in this graphic:

Abelian integral	**Berkovich-Coleman integral**
$\oplus$ path-independent	$\ominus$ path-dependent
$\oplus$ $[Q]-[P]$ torsion point of $J(\mathbb{C}_p)$ $\Longrightarrow$ ${}^{\mathrm{Ab}}\!\int_P^Q \omega = 0$	$\oplus$ better functorial properties
$\ominus$ hard to compute	$\oplus$ can be computed via primitives on open annuli

In particular the second property of the abelian integral is very interesting. It is a useful tool in the Chabauty-Coleman method, which is an effective method for bounding the number of rational points on a curve of genus $g \geq 2$ of small Mordell-Weil rank $\mathrm{rank}_{\mathbb{Z}}(J(\mathbb{Q})) < g$. Mostly it suffices to compute the integral for points P, Q lying in the same open ball isomorphic to $B(1)_+$, where it is possible to apply the Poincaré lemma. But sometimes one wants to calculate the integral for points that are not contained in the same open ball. Then it is almost impossible to calculate the abelian integral ${}^{\mathrm{Ab}}\!\int_P^Q \omega$. For the Berkovich-Coleman integral, on the contrary, computation is much easier. It cannot only be computed on open balls but also on open annuli isomorphic to $S(\rho)_+$, with $\rho \in |\mathbb{C}_p^\times|$. This creates much more possibilities. Even for two points not lying in the same open annulus there exists an algorithm of Balakrishnan, Bradshaw and Kedlaya (*Explicit Coleman Integration for Hyperelliptic Curves*, [BBK10]), which can be applied to hyperelliptic curves. To sum up, the computation of the Berkovich-Coleman integral

is in general not easy, but by far easier than computing the abelian integral if the points do not lie within the same open ball.
If one manages now two prove that both integrals coincide or that there is at least a rule how they differ, it would be possible to use the advantages of both integrals, especially the property

$$[Q]-[P] \text{ torsion point of } J(\mathbb{C}_p) \Longrightarrow {}^{\text{Ab}}\int_P^Q \omega = 0$$

of the abelian integral and the computation of the Berkovich-Coleman integral on open annuli. Michael Stoll proved exactly this. For $\dim_{\mathbb{C}_p}(\Omega^1_{X/\mathbb{C}_p}) > 1$, there exist differential one-forms $\omega \in \Omega^1_{X/\mathbb{C}_p}$ such that

$${}^{\text{Ab}}\int_P^Q \omega = {}^{\text{BC}}\int_P^Q \omega$$

on open annuli. Then, one could compute the Berkovich-Coleman integral and would simultaneously receive the abelian integral which can be used for the Chabauty-Coleman method.

1.3 Acknowledgements

Finally I would like to express my thanks to Prof. Dr. Walter Gubler for supervising my Master thesis. He spent a lot of time helping me when problems arised. In the beginning it was surely a big challenge to become confident with the theory that is required in the different integration theories but in the end understanding it and having accomplished this challenge makes me very happy. Special thanks go also to Florent Martin and Philipp Jell who made a great effort in answering my questions, and to Felix Herrmann for proofreading.

2 Basics

2.1 Berkovich analytic spaces

Let K be a field that is algebraically closed and complete with respect to a nontrivial, non-archimedean valuation $\mathrm{val} : K \longrightarrow \mathbb{R} \cup \{\infty\}$.

Definition 2.1.1. Let X be a topological Hausdorff space. A **quasi-net** on X is a family τ of compact subsets $V \subseteq X$ such that each $x \in X$ is contained in a closed neighbourhood of the form $V_1 \cup ... \cup V_n$, with $V_i \in \tau$ and $x \in V_1 \cap ... \cap V_n$.
A **net** is a quasi-net τ such that for all $V, V' \in \tau$ the family $\{W \in \tau : W \subseteq V \cap V'\}$ is a quasi-net on $V \cap V'$.

Example 2.1.2. (i) $\mathbb{R}^2$ is a topological Hausdorff space. The family

$$\tau = \left\{[i, i+1] \times [j, j+1] \subseteq \mathbb{R}^2 : i, j \in \mathbb{Z}\right\}$$

is a quasi-net on $\mathbb{R}^2$ because $\mathbb{R}^2$ is covered by these compact squares. But it is not a net, as e.g. for the intersection $([0,1]\times[0,1])\cap([1,2]\times[0,1]) = \{1\}\times[0,1]$, the family $\{W \in \tau : W \subseteq \{1\} \times [0,1]\}$ is empty and hence not a quasi-net on $\{1\} \times [0,1]$.

(ii) The family

$$\begin{aligned} \tau = &\left\{[i, i+1] \times [j, j+1] \subseteq \mathbb{R}^2 : i, j \in \mathbb{Z}\right\} \\ &\cup \left\{\{i\} \times [j, j+1] \subseteq \mathbb{R}^2 : i, j \in \mathbb{Z}\right\} \\ &\cup \left\{[i, i+1] \times \{j\} \subseteq \mathbb{R}^2 : i, j \in \mathbb{Z}\right\} \\ &\cup \left\{\{i\} \times \{j\} \subseteq \mathbb{R}^2 : i, j \in \mathbb{Z}\right\} \end{aligned}$$

is a net on $\mathbb{R}^2$. The intersection of $V, V' \in \tau$ is again in τ. So $V \cap V'$ is also contained in the family $\{W \in \tau : W \subseteq V \cap V'\}$ and hence the latter is a quasi-net as τ is already a quasi-net.

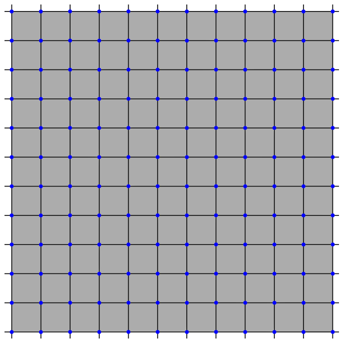


Figure 2.1: Net τ from (ii), without the edges and points it would be just a quasi-net

(iii) The family $\tau = \{\mathbb{R}^2\}$ is not a quasi-net on $\mathbb{R}^2$ because $\mathbb{R}^2$ is not compact.
(iv) $\tau = \{V \subseteq \mathbb{R}^2 : V \text{ closed and bounded}\}$ is a net as the intersection of two closed bounded sets is closed and bounded again.
(v) $\mathbb{C}_p$ is a topological Hausdorff space. It is not locally compact, so there exists an $x \in \mathbb{C}_p$ such that x has no compact neighbourhood. Therefore for any quasi-net τ on $\mathbb{C}_p$, the point x would not be contained in any element of τ. Hence there does not exist a quasi-net on $\mathbb{C}_p$.

Definition 2.1.3. Let X be a topological Hausdorff space together with a net τ. A **K-affinoid atlas** $\mathcal{A}$ on X with respect to τ is a family of affinoid algebras $\{\mathcal{A}_V\}_{V\in\tau}$, together with an isomorphism $V \stackrel{\sim}{\longrightarrow} \mathcal{M}(\mathcal{A}_V)$ for all $V \in \tau$, such that for all $V, V' \in \tau$ with $V' \subseteq V$, there exists a homomorphism of affinoid algebras $\mathcal{A}_V \longrightarrow \mathcal{A}_{V'}$ that makes V' to an affinoid subdomain of V.

Example 2.1.4. The space

$$(\mathbb{A}_K^n)^{\text{an}} = \{||.|| : K[T] \longrightarrow \mathbb{R}_{\geq} \text{ multiplicative seminorm that continues } |.|_K\}$$

is endowed with the weakest topology such that $||.|| \longmapsto ||f||$ is continuous for any $f \in K[T]$. One can show easily that this topological space is Hausdorff. Define

$$\tau := \left\{\mathcal{M}(T_{n,r}) \;\middle|\; r \in \mathbb{R}^n_{\geq 0}\right\}$$

where

$$T_{n,r} := \left\{ \sum_{I\in\mathbb{N}^n} a_I T^I \in K[[T]] \;\middle|\; |a_I| r^I \xrightarrow[|I|\to+\infty]{} 0 \right\}.$$

This is a net:

$$\begin{aligned}\mathcal{M}(T_{n,r}) = \{||.|| : T_{n,r} \longrightarrow \mathbb{R}_{\geq} &\text{ multiplicative seminorm} \\ &\text{that continues } |.|_K \text{ and is not zero}\}\end{aligned}$$

is a compact topological space (Satz 5.7, [Wer]) that can be considered as a subset of $(\mathbb{A}_K^n)^{\text{an}}$ by the restriction

$$\begin{aligned}\iota : \mathcal{M}(T_{n,r}) &\longrightarrow (\mathbb{A}_K^n)^{\text{an}}, \\ ||.|| &\longmapsto ||.|||_{K[T]}\end{aligned}$$

which is injective (Lemma 5.13, [Wer]). Hence τ consists of compact subsets and each $||.|| \in (\mathbb{A}_K^n)^{\text{an}}$ is contained in an $\mathcal{M}(T_{n,r})$ for a suitable $r \in \mathbb{R}^n_{\geq 0}$, as it is shown in the following:
Let $||.|| \in (\mathbb{A}_K^n)^{\text{an}}$ and define $r_1 := ||T_1||, ..., r_n := ||T_n||$. This allows to extend $||.||$ to a multiplicative seminorm $||.||_{\text{ext}}$ that continues $|.|_K$ and is not zero, such that $||.||_{\text{ext}} \in \mathcal{M}(T_{n,r})$ for $r := (r_1, ..., r_n)$ in the following way:

Let $f = \sum_{i\in\mathbb{N}^n} a_i T^i \in T_{n,r}$. By definition we have $|a_i|_K r^i \longrightarrow 0$ for $|i| \longrightarrow \infty$. Therefore the series $(f_k)_{k\in\mathbb{N}}$, defined by

$$f_k := \Big|\Big| \sum_{\substack{i\in\mathbb{N}^n \\ |i|\leq k}} a_i T^i \Big|\Big|,$$

converges in $\mathbb{R}_{\geq 0}$.

$$||f||_{\text{ext}} := \lim_{k\to+\infty} f_k$$

specifies a multiplicative seminorm on $T_{n,r}$ that extends $||.||$. Hence $\iota(||.||_{\text{ext}}) = ||.||$, and any $||.|| \in (\mathbb{A}^n_K)^{\text{an}}$ is contained in a closed neighbourhood $\mathcal{M}(T_{n,r})$ for a suitable $r \in \mathbb{R}^n_{\geq 0}$. Thus τ is a quasi-net. Since

$$\mathcal{M}(T_{n,r}) \cap \mathcal{M}(T_{n,s}) = \mathcal{M}(T_{n,t})$$

for $t \in \mathbb{R}^n_{\geq 0}$, defined by $t_i := \min\{r_i, t_i\}$, τ is a net on $(\mathbb{A}^n_K)^{\text{an}}$.
Then, by the definition $\mathcal{A}_{\mathcal{M}(T_{n,r})} := T_{n,r}$, the family

$$\mathcal{A} := \{\mathcal{A}_V\}_{V\in\tau}$$

is a K-affinoid atlas with respect to τ, as the trivial isomorphism

$$\mathcal{M}(T_{n,r}) \xrightarrow{\sim} \mathcal{M}(\mathcal{A}_V) = \mathcal{M}(T_{n,r}),$$

exists and for $\mathcal{M}(T_{n,r}) \subseteq \mathcal{M}(T_{n,s})$, that is $r_i \leq s_i$ for all $i \in \{1, ..., n\}$, the inclusion

$$T_{n,s} \longrightarrow T_{n,r}$$

constitutes $\mathcal{M}(T_{n,r})$ an affinoid subdomain of $\mathcal{M}(T_{n,s})$.

Definition 2.1.5. A ***K*-analytic space**, in the sense of Berkovich, is a tuple $(X, \tau, \mathcal{A})$ where X is a topological Hausdorff space, τ a net on X and $\mathcal{A}$ a K-affinoid atlas with respect to τ.

Remark 2.1.6. Let $(X, \tau_1, \mathcal{A}_1)$ and $(X, \tau_2, \mathcal{A}_2)$ be two K-analytic spaces with the same underlying topological Hausdorff space X. If $\tau_1 \subseteq \tau_2$ and correspondingly $\mathcal{A}_1 \subseteq \mathcal{A}_2$ such that $(\mathcal{A}_1)_V = (\mathcal{A}_2)_V$ for $V \in \tau_1$, one says the K-analytic space $(X, \tau_2, \mathcal{A}_2)$ refines $(X, \tau_1, \mathcal{A}_1)$ and one identifies both with each other.

Example 2.1.7. $((\mathbb{A}^n_K)^{\text{an}}, \tau, \mathcal{A})$ from Example 2.1.4 is a K-analytic space.

Remark 2.1.8. By section 3.1, [Ber09], the K-analytic spaces form a category, meaning that morphisms between K-analytic spaces are well-defined.

Definition 2.1.9. A morphism of K-affinoid spaces $\varphi : \mathcal{M}(\mathcal{A}) \longrightarrow \mathcal{M}(\mathcal{B})$ is **finite** if the canonical homomorphism $\mathcal{B} \longrightarrow \mathcal{A}$ of K-affinoid algebras makes $\mathcal{A}$ to a finite Banach $\mathcal{B}$-algebra.
A morphism of K-analytic spaces $\varphi : X \longrightarrow Y$ is **finite** if there exists a family of affinoid domains $\{V_i\}_{i\in I}$ which is a covering of Y such that all $\varphi^{-1}(V_i) \longrightarrow V_i$ are finite morphisms of K-affinoid spaces.

Definition 2.1.10. Let X, Y be K-analytic spaces. A morphism $\varphi : X \longrightarrow Y$ is said to be **finite étale** if it is finite and it fulfils the following property: For any affinoid domain $V = \mathcal{M}(\mathcal{B}) \subseteq Y$ with $\varphi^{-1}(V) = \mathcal{M}(\mathcal{A})$, where $\mathcal{A}$ and $\mathcal{B}$ are K-affinoid algebras, $\mathcal{A}$ is a finite étale $\mathcal{B}$-algebra.
φ is said to be **étale** if, for any point $x \in X$, there exists an open neighbourhood $x \in U \subseteq X$ and $\varphi(x) \in V \subseteq Y$ such that φ induces a finite étale morphism $U \longrightarrow V$.

Example 2.1.11. Let X, Y be K-analytic spaces. If the morphism $\varphi : X \longrightarrow Y$ is an open immersion, it is étale.

Definition 2.1.12. Let X, Y be K-analytic spaces. A morphism $\varphi : X \longrightarrow Y$ is **smooth at a point** $x \in X$ if there exists an open neighbourhood $x \in U \subseteq X$ such that the induced morphism $U \longrightarrow Y$ factorizes as

$$\begin{tikzcd} U \arrow[rr, "\varphi|_U"] \arrow[dr, "\varphi_1"'] & & Y \\ & Y \times (\mathbb{A}^n_K)^{\mathrm{an}} \arrow[ur, "\varphi_2"'] & \end{tikzcd}$$

where φ_1 is an étale morphism and φ_2 is the canonical projection. φ is said to be **smooth** if it is smooth at all points $x \in X$.

Definition 2.1.13. A K-analytic space X is **smooth** if the canonical morphism $X \longrightarrow \mathcal{M}(K)$ is smooth.

Example 2.1.14. $((\mathbb{A}^n_K)^{\mathrm{an}}, \tau, \mathcal{A})$ from Example 2.1.4 is a smooth K-analytic space.

Proof. It suffices to prove that the canonical morphism

$$(\mathbb{A}^n_K)^{\mathrm{an}} \longrightarrow \mathcal{M}(K) = \{|.|_K\}$$

is smooth. Considering the trivial open neighbourhood $U = (\mathbb{A}^n_K)^{\mathrm{an}}$, the diagram

$$\begin{tikzcd} (\mathbb{A}^n_K)^{\mathrm{an}} \arrow[rr] \arrow[dr, "\varphi_1"'] & & \{|.|_K\} \\ & \{|.|_K\} \times (\mathbb{A}^n_K)^{\mathrm{an}} . \arrow[ur, "\varphi_2"'] & \end{tikzcd}$$

commutes with $\varphi_1 : ||.|| \longmapsto (|.|_K, ||.||)$ and $\varphi_2 : (|.|_K, ||.||) \longmapsto |.|_K$. Since φ_1 can be identified with the identity map on $(\mathbb{A}^n_K)^{\mathrm{an}}$, which is étale, $((\mathbb{A}^n_K)^{\mathrm{an}}, \tau, \mathcal{A})$ is a smooth K-analytic space. $\square$

As it always holds $Y = \mathcal{M}(K)$ in the definition of a smooth K-analytic space, it is possible to simplify the definition of a K-analytic space:

Lemma 2.1.15. *A K-analytic space X is smooth if and only if for any $x \in X$ there exists an open neighbourhood $x \in U \subseteq X$ such that there is an étale morphism*

$$U \longrightarrow (\mathbb{A}_K^n)^{\mathrm{an}}$$

for some $n \in \mathbb{N}\backslash\{0\}$.

Proof. With $Y = \mathcal{M}(K) = \{|.|_K\}$, the diagram of Definition 2.1.12 becomes

$$\begin{tikzcd} U \arrow[rr, "\varphi|_U"] \arrow[dr, "\varphi_1"'] & & \{|.|_K\} \\ & \{|.|_K\} \times (\mathbb{A}_K^n)^{\mathrm{an}}, \arrow[ur, "\varphi_2"'] & \end{tikzcd}$$

which always commutes if the étale morphism φ_1 exists because $\varphi|_U$ and φ_2 are trivial. Eventually $\{|.|_K\} \times (\mathbb{A}_K^n)^{\mathrm{an}}$ can be identified with $(\mathbb{A}_K^n)^{\mathrm{an}}$. □

Example 2.1.16. By Example 2.1.11 and Lemma 2.1.15, every K-analytic space, which locally has an open immersion into $(\mathbb{A}_K^n)^{\mathrm{an}}$ for some $n \in \mathbb{N}\backslash\{0\}$, is smooth.

Remark 2.1.17. Berkovich has shown that there exists a so called GAGA functor that associates to every K-scheme X, locally of finite type, a K-analytic space X^{an}, which is called the analytification of X. Especially the analytification of an affine K-scheme $X = \mathrm{Spec}(A)$, locally of finite type, where A is a finitely generated ring over K, is equal to the set of all multiplicative seminorms $\mathcal{M}(A)$ on A that continues the absolute value on K.

Lemma 2.1.18. *For the analytification X^{an} of a K-scheme X that is locally of finite type, it holds*

(i) X is separated if and only if X^{an} is Hausdorff,
(ii) X is proper if and only if X^{an} is compact and Hausdorff,
(iii) X is connected if and only if X^{an} is path-connected,
(iv) X is smooth if and only if X^{an} is smooth.

Here just the underlying topological space of X^{an} is considered.

Proof. (i) Theorem 3.4.8 (i), [Ber90].
(ii) Theorem 3.4.8 (ii), [Ber90].
(iii) Theorem 3.4.8 (iii), [Ber90].
(iv) Proposition 3.4.3, [Ber90]. □

Example 2.1.19. The analytification of the parabola

$$\mathrm{Spec}(K[T_1, T_2]/(T_2 - T_1^2))$$

is a smooth K-analytic space.

Proof. By Remark 2.1.17, the analytification of $\operatorname{Spec}(K[T_1, T_2]/(T_2 - T_1^2))$ is

$$\mathcal{M}(K[T_1, T_2]/(T_2 - T_1^2)).$$

Since

$$K[T_1, T_2]/(T_2 - T_1^2) \cong K[T_1]$$

holds, $K[T_1, T_2]/(T_2 - T_1^2)$ is a finite étale $K[T_1]$-algebra. Consequently

$$\mathcal{M}(K[T_1, T_2]/(T_2 - T_1^2)) \longrightarrow \mathcal{M}(K[T_1]) = (\mathbb{A}_K^1)^{\mathrm{an}}$$

is a finite étale morphism, as it is finite, and therefore étale at any point of

$$\mathcal{M}(K[T_1, T_2]/(T_2 - {T_1}^2)).$$

Lemma 2.1.15 yields that

$$\mathcal{M}(K[T_1, T_2]/(T_2 - T_1^2)) = \left(\operatorname{Spec}(K[T_1, T_2]/(T_2 - T_1^2))\right)^{\mathrm{an}}$$

is smooth. □

Example 2.1.20. The analytification of the cross

$$\operatorname{Spec}(K[T_1, T_2]/(T_1 T_2))$$

is not a smooth K-analytic space.

Proof. By Remark 2.1.17, the analytification of $\operatorname{Spec}(K[T_1, T_2]/(T_1 T_2))$ is

$$\mathcal{M}(K[T_1, T_2]/(T_1 T_2)).$$

The aim is to show that the analytic space is not smooth at $(0, 0)$. Let U be a neighbourhood of 0. Then

$$U = \mathcal{M}\left(\left\{\overline{f} \in K\{T_1, T_2\}/(T_1 T_2) : f \text{ converges on } U\right\}\right),$$

where f converges on U means that $||f||$ is well-defined for all $||.|| \in U$, and $K\{T_1, T_2\}$ denotes the power series in T_1 and T_2.
For smoothness on U it would be necessary to show that

$$U = \mathcal{M}\left(\left\{\overline{f} \in K\{T_1, T_2\}/(T_1 T_2) : f \text{ converges on } U\right\}\right) \longrightarrow \mathcal{M}(K[T_1, ..., T_n]) = (\mathbb{A}_K^n)^{\mathrm{an}}$$

is étale for some $n \in \mathbb{N}\backslash\{0\}$, which is equivalent to verifying

$$\left\{\overline{f} \in K\{T_1, T_2\}/(T_1 T_2) : f \text{ converges on } U\right\}$$

is a finite étale $K[T_1, ..., T_n]$-algebra. For $n = 1$ it is not finite. Taking $n > 2$, it is not a $K[T_1, ..., T_n]$-algebra anymore. The only remaining possibility is $n = 2$. The latter is equivalent to

$$\mathrm{Spec}\left(\left\{\overline{f} \in K\{T_1, T_2\}/(T_1T_2) : f \text{ converges on } U\right\}\right) \longrightarrow \mathrm{Spec}(K[T_1, T_2]) \tag{2.1}$$

being an étale morphism of schemes. By Theorem 5.1, Exposé I, [GR02], a morphism of schemes that is of finite type, such as (2.1), is an open immersion if and only if it is universally injective and étale. A morphism $X \longrightarrow S$ is universally injective if and only if for any morphism of schemes $S' \longrightarrow S$ the base change $X_{S'} \longrightarrow S'$ is injective. Since

$$\mathrm{Spec}\left(\left\{\overline{f} \in K\{T_1, T_2\}/(T_1T_2) : f \text{ converges on } U\right\}\right)$$

is just a subset of $\mathrm{Spec}(K[T_1, T_2])$, it is universally injective. Therefore the morphism (2.1) is étale if and only if it is an open immersion. But

$$\mathrm{Spec}\left(\left\{\overline{f} \in K\{T_1, T_2\}/(T_1T_2) : f \text{ converges on } U\right\}\right)$$

is not open in $\mathrm{Spec}(K[T_1, T_2])$. Ergo, there does not exist a neighbourhood U of $(0, 0)$ and an $n \in \mathbb{N}\backslash\{0\}$ such that

$$U \longrightarrow (\mathbb{A}^n_K)^{\mathrm{an}}$$

is étale. Hence $\mathcal{M}(K[T_1, T_2]/(T_1T_2))$ is not smooth at $(0, 0)$. □

Example 2.1.21. The analytification of the hyperbola

$$\mathrm{Spec}(K[T_1, T_2]/(T_1T_2 - \varpi))$$

is a smooth K-analytic space for $\varpi \in K^\times$.

Proof. By Remark 2.1.17, the analytification of $\mathrm{Spec}(K[T_1, T_2]/(T_1T_2 - \varpi))$ is

$$\mathcal{M}(K[T_1, T_2]/(T_1T_2 - \varpi)).$$

Considering an element

$$||.|| \in \mathcal{M}(K[T_1, T_2]/(T_1T_2 - \varpi)),$$

it holds

$$||T_1T_2 - \varpi|| = 0,$$

and hence $||T_1|| \neq 0$ because otherwise $||T_1T_2|| = 0$ and then

$$||T_1T_2 - \varpi|| = |\varpi| \neq 0$$

would hold. It follows that

$$||\tfrac{1}{T_1}|| = \tfrac{1}{||T_1||}$$

is well-defined. Thus the evaluation morphism

$$T_2 \longmapsto \frac{\varpi}{T_1}$$

delivers the isomorphism

$$K[T_1, T_2]/(T_1T_2 - \varpi) \cong K[T_1, T_1^{-1}]$$

which yields that

$$\mathcal{M}(K[T_1, T_2]/(T_1T_2 - \varpi)) \longrightarrow \mathcal{M}(K[T_1, T_1^{-1}]) = (T_K^1)^{\mathrm{an}}$$

is a finite étale morphism. Furthermore the injection of the analytification of the one-dimensional torus

$$(T_K^1)^{\mathrm{an}} \hookrightarrow (\mathbb{A}_K^1)^{\mathrm{an}}$$

is étale as it is an open immersion. Since a composition of étale morphisms is still étale,

$$\mathcal{M}(K[T_1, T_2]/(T_1T_2 - \varpi)) \longrightarrow \mathcal{M}(K[T_1]) = (\mathbb{A}_K^1)^{\mathrm{an}}$$

is étale, meaning that the hyperbola is smooth. □

Remark 2.1.22. The three examples comply with property (iv) of Lemma 2.1.18. As K-schemes, all examples are smooth, apart from the cross, which is not smooth at $(0, 0)$.

2.2 p-adic Lie theory

Let K be a field that is algebraically closed and complete with respect to a nontrivial, non-archimedean valuation $\mathrm{val} : K \longrightarrow \mathbb{R} \cup \{\infty\}$.

2.2.1 Analytic manifolds

Let M be a Hausdorff topological space, $U \subseteq K^r$ an open subset and E a K-Banach space.

Definition 2.2.1. A function $f : U \longrightarrow E$ is called **analytic** if, for any point $x_0 \in U$, there is an open ball $B_\epsilon(x_0) \subseteq U$, and f is equal to a power series convergent on this open ball.

Definition 2.2.2. A **chart** for M is a triple (U, φ, K^n) consisting of an open subset $U \subseteq M$ and a map $\varphi : U \longrightarrow K^n$ such that:

(i) $\varphi(U)$ is open in K^n
(ii) $\varphi : U \xrightarrow{\cong} \varphi(U)$ is a homeomorphism.

Definition 2.2.3. Two charts $(U_1, \varphi_1, K^{n_1})$ and $(U_2, \varphi_2, K^{n_2})$ are called **compatible** if both maps

$$\begin{tikzcd} \varphi_1(U_1 \cap U_2) \arrow[r, shift left, "{\varphi_2 \circ \varphi_1}^{-1}"] & \varphi_1(U_1 \cap U_2) \arrow[l, shift left, "{\varphi_1 \circ \varphi_2}^{-1}"] \end{tikzcd}$$

are analytic.

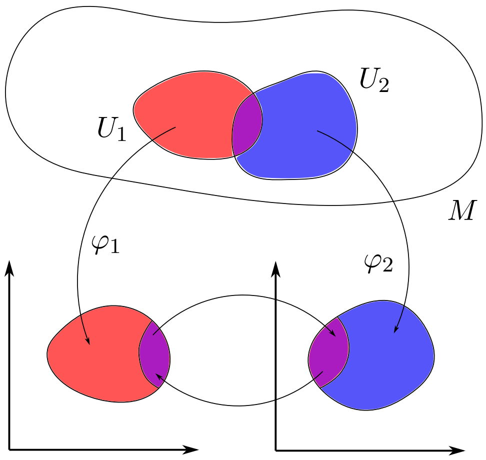


Figure 2.2: Compatible charts if the lower two maps are analytic

Definition 2.2.4. An **atlas** for M is a set $\mathcal{A} = \{(U_i, \varphi_i, K^n)\}_{i \in I}$ of charts for M such that

(i) any two of these charts are compatible and
(ii) $M = \bigcup_{i \in I} U_i$.

Definition 2.2.5. Two atlases $\mathcal{A}$ and $\mathcal{B}$ for M are called **equivalent** if $\mathcal{A} \cup \mathcal{B}$ is also an atlas for M.

Definition 2.2.6. An atlas $\mathcal{A}$ for M is called **maximal** if any equivalent atlas $\mathcal{B}$ for M satisfies $\mathcal{B} \subseteq \mathcal{A}$.

Definition 2.2.7. A **naive K-analytic manifold** $(M, \mathcal{A})$ is a Hausdorff topological space M, equipped with a maximal atlas $\mathcal{A}$.

Lemma 2.2.8. *Let $(M, \mathcal{A})$ be a naive K-analytic manifold and $U \subseteq M$ an open subset. Then*

$$\mathcal{A}_U := \{(V, \psi, K^n) \in \mathcal{A} | V \subseteq U\}$$

is a maximal atlas for U.

Proof. The claim after Remark 8.1, [Sch11]. □

Definition 2.2.9. $(U, \mathcal{A}_U)$ from Lemma 2.2.8 is called a **naive open K-analytic submanifold** of $(M, \mathcal{A})$.

Example 2.2.10. (i) K^n with

$$\mathcal{A} = \{(U_i, \varphi_i, K^n) |\ U_i \subseteq K^n \text{ open},\ \varphi_i = \mathrm{id}\}_{i \in I}$$

is a naive K-analytic manifold. Analogous $(K^\times)^n$.

(ii) $\mathbb{C}_p$ is a naive $\mathbb{C}_p$-analytic manifold as $\mathbb{C}_p$ is open in $\mathbb{C}_p$, and therefore it is a naive open $\mathbb{C}_p$-analytic submanifold.

Definition 2.2.11. A function $f : M \longrightarrow E$ is called **analytic** if

$$f \circ \varphi^{-1} : \varphi(U) \longrightarrow E$$

is an analytic function for any chart (U, φ) of M.

Let N be a naive K-analytic manifold, too.

Definition 2.2.12. A function $g : M \longrightarrow N$ is called **analytic** if, for any point $x \in M$, there exists a chart (U, φ, K^m) for M around x and a chart (V, ψ, K^n) for N around $g(x)$, such that

(i) $g(U) \subseteq V$ and
(ii) $\psi \circ g \circ \varphi^{-1} : \varphi(U) \longrightarrow K^n$ is analytic.

2.2.2 Tangent spaces

Let M be a naive K-analytic manifold.

Definition 2.2.13. Fix $a \in M$. Let $c = (U, \varphi, K^m)$ be a chart for M around a. Furthermore $v \in K^m$. Two pairs (c, v) and (c', v') are called **equivalent** if

$$\mathrm{d}_{\varphi(a)}(\varphi' \circ \varphi^{-1})(v) = v'.$$

Definition 2.2.14. A **tangent vector** of M at the point a is an equivalence class $[c, v]$ of pairs (c, v).

Definition 2.2.15. The **tangent space** of M at a is defined as

$$T_a(M) := \{\text{tangent vectors of } M \text{ at } a\}.$$

Lemma 2.2.16. *Let $g : M \longrightarrow N$ be an analytic map of naive K-analytic manifolds. By Definition 2.2.12, there exist a chart $c = (U, \varphi, K^m)$ for M around x and a chart $\tilde{c} = (V, \psi, K^n)$ for N around $g(x)$, such that $g(U) \subseteq V$. Furthermore there is the map*

$$\theta_c : K^m \longrightarrow T_a(M), v \longmapsto [c, v].$$

Then

$$T_a(g) :\ T_a(M) \xrightarrow{\theta_c{}^{-1}} K^m \xrightarrow{D_{\varphi(a)}(\psi \circ g \circ \varphi^{-1})} K^n \xrightarrow{\theta_{\tilde{c}}} T_{g(a)}(N)$$

is a continuous K-linear map which is independent of the choice of charts.

Proof. The paragraph after the first Remark in section 9, [Sch11]. □

Definition 2.2.17. $T_a(g)$, from Lemma 2.2.16, is called the **tangent map** of g at the point a.

Lemma 2.2.18. *(i)* $T_a(\mathrm{id}_M) = \mathrm{id}_{T_a(M)}$

(ii) Let $L \xrightarrow{f} M \xrightarrow{g} N$ *be two maps of naive* K*-analytic manifolds. Then*

$$T_a(g \circ f) = T_{f(a)}(g) \circ T_a(f)$$

holds for any $a \in L$.

Proof. Lemma 9.2 and the Remark before, [Sch11]. □

2.2.3 Lie groups

Definition 2.2.19. A K**-Lie group** G is a naive K-analytic manifold which carries the structure of a group such that the multiplication map

$$m : G \times G \longrightarrow G, (g, h) \longmapsto gh$$

is analytic.

Remark 2.2.20. Instead of a C^∞-manifold, which is required for defining common Lie groups, here a naive K-analytic manifold is used. From now on, whenever Lie groups are mentioned, we are talking about K-Lie groups.

Example 2.2.21. (i) K^n is a K-Lie group.

Proof. (a) Naive K-analytic manifold: Example 2.2.10
(b) Group structure: Addition
(c) Analytic group multiplication:

$$m : K^n \times K^n \longrightarrow K^n, (a, b) \longmapsto a + b$$

can be written as

$$m : K^{2n} \longrightarrow K^n, x \longmapsto \begin{pmatrix} x_1 + x_{n+1} \\ x_2 + x_{n+2} \\ \ldots \\ x_n + x_{2n} \end{pmatrix},$$

where the entries are polynomials, and m is therefore analytic.

□

(ii) $B_\epsilon^+(0) = \{x \in K^n | |x| \leq \epsilon\}$, with $\epsilon > 0$, is a K-Lie group.

Proof. (a) Naive K-analytic manifold: Naive open K-analytic submanifold of K^n

(b) Group structure: Subgroup of K^n with addition, since

$$|a+b| \leq \max(|a|,|b|) \leq \epsilon$$

(c) Analytic group multiplication: See (i)

$\square$

(iii) $\{x \in \mathbb{C}_p | \; |x| = 1\}$ is a $\mathbb{C}_p$-Lie group.

Proof. (a) Naive $\mathbb{C}_p$-analytic manifold: Naive open $\mathbb{C}_p$-analytic submanifold of $\mathbb{C}_p$
(b) Group structure: Subgroup of $\mathbb{C}_p$ with multiplication, since $|a \cdot b| = |a| \cdot |b| = 1$
(c) Analytic group multiplication:

$$m : \mathbb{C}_p \times \mathbb{C}_p \longrightarrow \mathbb{C}_p, (a,b) \longmapsto a \cdot b$$

is a polynomial, and m is therefore analytic.

$\square$

(iv) $\mathrm{GL}_n(K)$ is a K-Lie group.

Proof. (a) Naive K-analytic manifold: Naive open K-analytic submanifold of K^{n^2}, since $\det : M_n(K) \longrightarrow K$ is continuous and $K^\times$ is open in K.
(b) Group structure: Clear
(c) Analytic group multiplication: As every entry $c_{ik} = \sum_{j=1}^n a_{ij} \cdot b_{jk}$ of the matrix product $A \cdot B$ is a polynomial, the multiplication is analytic.

$\square$

Definition 2.2.22. Let G, H be K-Lie groups. The map

$$\varphi : G \longrightarrow H$$

is a **K-Lie group homomorphism** if

(i) φ is a group homomorphism of the underlying groups and
(ii) φ is an analytic function of the underlying naive K-analytic manifolds.

2.2.4 Lie algebras

Definition 2.2.23. A **Lie algebra** is a K-vector space $\mathfrak{g}$ together with the operation

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}, \quad (x,y) \longmapsto [x,y]$$

such that the following conditions hold:

(i) $[\cdot,\cdot]$ is bilinear.
(ii) The Jacobi identity $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$ holds.
(iii) $[x,x] = 0$ for every $x \in \mathfrak{g}$.

$[\cdot,\cdot]$ is called the Lie bracket.

Definition 2.2.24. Denote the identity element of G by e. Then

$$\mathrm{Lie}(G) := (T_e(G), [\cdot,\cdot])$$

is called the **Lie algebra of** G. For the definition of the brackets see section 13, [Sch11].

2.3 Weighted graphs

Definition 2.3.1. A **graph** G is defined as a non-empty set of vertices $V(G)$ and a set of edges $E(G)$ together with an edge assignment map

$$\iota_{\text{ass}} : E(G) \longrightarrow V(G) \times V(G).$$

For an edge $e \in E(G)$ with $\iota_{\text{ass}}(e) = (V_1, V_2)$, the element $e^- := V_1$ is said to be the tail vertex and $e^+ := V_2$ the head vertex of e.

Definition 2.3.2. A graph G is called **connected** if for any two vertices $V, V' \in V(G)$ there exists a finite number of vertices $V_0 = V, V_1, ..., V_{n-1}, V_n = V'$ and a finite number of edges $e_1, ..., e_n \in E(G)$ such that

$$\iota_{\text{ass}}(e_i) = (V_{i-1}, V_i).$$

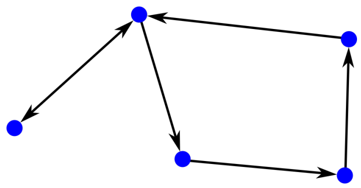


Figure 2.3: Connected graph

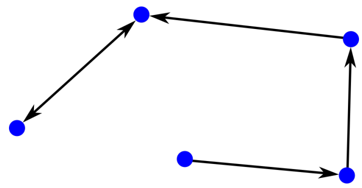


Figure 2.4: Disconnected graph

Definition 2.3.3. A graph G is called **antisymmetric** if for any two edges e, e', with $\iota_{\text{ass}}(e) = (V_1, V_2)$ and $\iota_{\text{ass}}(e') = (V_2, V_1)$, it holds $V_1 = V_2$.

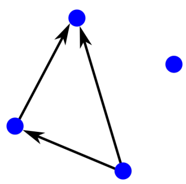


Figure 2.5: Antisymmetric graph

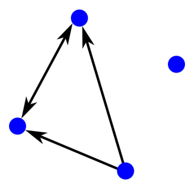


Figure 2.6: Non-antisymmetric graph

Definition 2.3.4. A graph G is called **weighted** if there exists a length map

$$l : E(G) \longrightarrow \mathbb{R}_{>0}.$$

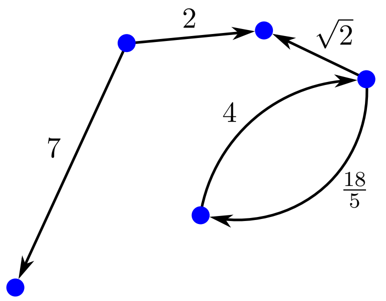


Figure 2.7: Weighted graph

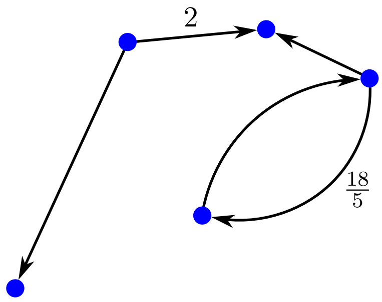


Figure 2.8: Not a weighted graph

Definition 2.3.5. Let G be a connected, antisymmetric, weighted graph and $A = \mathbb{Z}$ or $A = \mathbb{R}$. Define $C_0(G, A)$ to be the free A-module generated by the vertex set $V(G)$. Furthermore $C_1(G, A)$ is defined to be the free A-module generated by the edge set $E(G)$. The elements of $C_0(G, A)$ are called 0-chains with coefficients in A, and the elements of $C_1(G, A)$ are called 1-chains with coefficients in A.

Let G be a connected, antisymmetric, weighted graph for the rest of the section.

Remark 2.3.6. For $e \in E(G)$ one may consider the element $(-1)e \in C_1(G, A)$ as the inverse edge, that is the edge with changed tail and head vertex.

Remark 2.3.7. As any element $f = \sum_{V \in V(G)} n_V V \in C_0(G, A)$ is a finite A-linear combination of vertices, meaning $n_V \in A$, and $n_V \neq 0$ just for a finite number of $V \in V(G)$, one may identify f with the function

$$\begin{aligned} f : V(G) &\longrightarrow A, \\ V &\longmapsto n_V. \end{aligned}$$

Doing the same for any $\alpha = \sum_{e \in E(G)} n_e e \in C_1(G, A)$, allows to identify α with the function

$$\begin{aligned} \alpha : E(G) &\longrightarrow A, \\ e &\longmapsto n_e. \end{aligned}$$

Definition 2.3.8. Define the following products,

$$\langle f_1, f_2 \rangle_0 := \sum_{V \in V(G)} f_1(V) f_2(V)$$

for $f_1, f_2 \in C_0(G, \mathbb{R})$ and

$$\langle \alpha_1, \alpha_2 \rangle_1 := \sum_{e \in E(G)} \alpha_1(e) \alpha_2(e) l(e)$$

for $\alpha_1, \alpha_2 \in C_1(G, \mathbb{R})$, on $C_0(G, \mathbb{R})$ respectively $C_1(G, \mathbb{R})$.

Lemma 2.3.9. *Both products from Definition 2.3.8 are scalar products.*

Proof. Since $A = \mathbb{R}$ is a field, the A-module $C_0(G, A)$ respectively $C_1(G, A)$ becomes an $\mathbb{R}$-vector space $C_0(G, \mathbb{R})$ respectively $C_1(G, \mathbb{R})$, where a scalar product can be defined.
Linearity: Both products are linear in both arguments.
Symmetry: Follows from the commutativity of $\mathbb{R}$.
Positive definiteness: It holds

$$\langle f, f \rangle_0 := \sum_{V \in V(G)} (f(V))^2 \geq 0$$

for $f \in C_0(G, \mathbb{R})$, and $\sum_{V \in V(G)} (f(V))^2 = 0$ only if $f(V) = 0$ for all $V \in V(G)$, which means $f = 0$. The same result follows for the product on $C_1(G, \mathbb{R})$, as $l(e) \in \mathbb{R}_{>0}$ for all $e \in C_1(G, \mathbb{R})$. $\square$

Definition 2.3.10. The **differential operator** d on $C_0(G, \mathbb{R})$ is defined as the map

$$\begin{aligned} \mathrm{d}\colon C_0(G, \mathbb{R}) &\longrightarrow C_1(G, \mathbb{R}), \\ f &\longmapsto \mathrm{d}f \end{aligned}$$

with $\mathrm{d}f \in C_1(G, \mathbb{R})$ specified as

$$\begin{aligned} \mathrm{d}f\colon E(G) &\longrightarrow \mathbb{R}, \\ e &\longmapsto \frac{f(e^+) - f(e^-)}{l(e)}. \end{aligned}$$

Lemma 2.3.11. *The adjoint operator of the differential operator* d*, with respect to the two scalar products from Definition 2.3.8, is given by*

$$\begin{aligned} \mathrm{d}^*\colon C_1(G, \mathbb{R}) &\longrightarrow C_0(G, \mathbb{R}), \\ \alpha &\longmapsto \mathrm{d}^*\alpha \end{aligned}$$

with $\mathrm{d}^*\alpha \in C_0(G, \mathbb{R})$ *defined as*

$$\begin{aligned} \mathrm{d}^*\alpha : V(G) &\longrightarrow \mathbb{R}, \\ V &\longmapsto \sum_{\substack{e \in E(G) \\ e^+ = V}} \alpha(e) - \sum_{\substack{e \in E(G) \\ e^- = V}} \alpha(e). \end{aligned}$$

Proof. First, the adjoint operator of d exists and is unique as d is linear. So it suffices to prove the equality

$$\langle \mathrm{d}f, \alpha \rangle_1 = \langle f, \mathrm{d}^*\alpha \rangle_0$$

for all $f \in C_0(G, \mathbb{R}), \alpha \in C_1(G, \mathbb{R})$:

$$
\begin{aligned}
\langle \mathrm{d}f, \alpha \rangle_1 &= \sum_{e \in E(G)} \mathrm{d}f(e)\alpha(e)l(e) \\
&= \sum_{e \in E(G)} \frac{f(e^+) - f(e^-)}{l(e)} \alpha(e)l(e) \\
&= \sum_{e \in E(G)} \left(f(e^+) - f(e^-) \right) \alpha(e) \\
&= \sum_{e \in E(G)} f(e^+)\alpha(e) - \sum_{e \in E(G)} f(e^-)\alpha(e) \\
&= \sum_{V \in V(G)} \left(\sum_{\substack{e \in E(G) \\ e^+ = V}} f(V)\alpha(e) - \sum_{\substack{e \in E(G) \\ e^- = V}} f(V)\alpha(e) \right) \\
&= \sum_{V \in V(G)} f(V) \left(\sum_{\substack{e \in E(G) \\ e^+ = V}} \alpha(e) - \sum_{\substack{e \in E(G) \\ e^- = V}} \alpha(e) \right) \\
&= \sum_{V \in V(G)} f(V)\, \mathrm{d}^*\alpha(V) \\
&= \langle f, \mathrm{d}^*\alpha \rangle_0
\end{aligned}
$$

□

Definition 2.3.12. The **real 1-cycles** are defined as $H_1(G, \mathbb{R}) := \ker(\mathrm{d}^*)$ and the **integral 1-cycles** as $H_1(G, \mathbb{Z}) := H_1(G, \mathbb{R}) \cap C_1(G, \mathbb{Z})$.

Example 2.3.13. The orange 1-chain $e_3 + e_2 - e_1$ in Figure 2.9 is a cycle, but the green 1-chain $e_5 + e_6$ not because it has the vertex at the left top just once as a tail vertex and not as a head vertex.

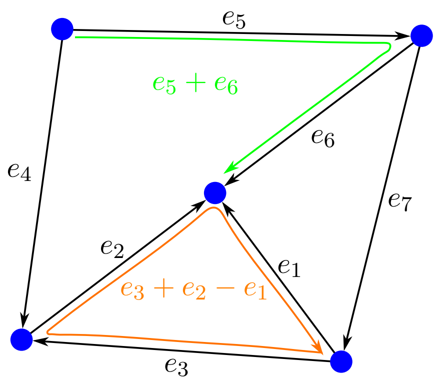


Figure 2.9: 1-cycles form loops

Definition 2.3.14. Let G_1, G_2 be two connected, antisymmetric, weighted graphs. It is said that G_2 **refines** G_1 if there exists an injection

$$a : V(G_1) \hookrightarrow V(G_2)$$

and a surjection

$$b : E(G_2) \twoheadrightarrow E(G_1)$$

such that for any edge of G_1 there exist vertices

$$V_0 = a(e^-), V_1, ..., V_{n-1}, V_n = a(e^+) \in V(G_2)$$

and edges

$$e_1, ..., e_n \in E(G_2)$$

such that

(i) $b^{-1}(e) = \{e_1, ..., e_n\}$,
(ii) $\sum_{i=1}^{n} l(e_i) = l(e)$ and
(iii) $\iota_{\text{ass}}(e_i) = (V_{i-1}, V_i)$ for all $i = 1, ..., n$

where ι_{ass} is the edge assignment map with respect to G_2.
For $\alpha = \sum_{e \in E(G_1)} n_e e \in C_1(G_1, A)$ the element

$$\alpha_{\text{refine}} = \sum_{e \in E(G_2)} n_{b(e)} e \in C_1(G_2, A)$$

is called the **refinement** of α.

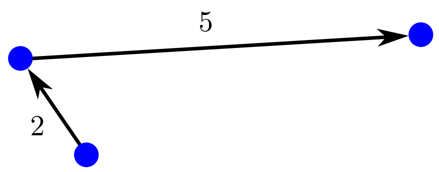


Figure 2.10: Graph G

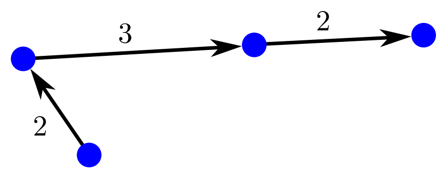


Figure 2.11: This graph refines G

Definition 2.3.15. Define the **edge length pairing** as the map

$$\begin{aligned}
[.,.] : C_1(G, A) \times C_1(G, A) &\longrightarrow \mathbb{R}, \\
(\alpha_1, \alpha_2) &\longmapsto \sum_{e \in E(G)} n_e m_e l(e)
\end{aligned}$$

with $\alpha_1 = \sum_{e \in E(G)} n_e e,\ \alpha_2 = \sum_{e \in E(G)} m_e e \in C_1(G, A)$.

Lemma 2.3.16. *The edge length pairing is bilinear and symmetric.*

Proof. Symmetry follows directly from the commutativity of $\mathbb{R}$. For $\lambda \in A$ and $\alpha_1 = \sum_{e\in E(G)} n_e e,\ \alpha_1' = \sum_{e\in E(G)} n_e' e,\ \alpha_2 = \sum_{e\in E(G)} m_e e \in C_1(G, A)$ we have

$$\begin{aligned}
[\lambda\alpha_1, \alpha_2] &= \left[\lambda \sum_{e\in E(G)} n_e e, \sum_{e\in E(G)} m_e e\right] \\
&= \left[\sum_{e\in E(G)} \lambda n_e e, \sum_{e\in E(G)} m_e e\right] \\
&= \sum_{e\in E(G)} \lambda n_e m_e l(e) \\
&= \lambda \sum_{e\in E(G)} n_e m_e l(e) \\
&= \lambda\,[\alpha_1, \alpha_2]
\end{aligned}$$

and

$$\begin{aligned}
\left[\alpha_1 + \alpha_1', \alpha_2\right] &= \left[\left(\sum_{e\in E(G)} n_e e + \sum_{e\in E(G)} n_e' e\right), \sum_{e\in E(G)} m_e e\right] \\
&= \left[\sum_{e\in E(G)} \left(n_e + n_e'\right) e, \sum_{e\in E(G)} m_e e\right] \\
&= \sum_{e\in E(G)} \left(n_e + n_e'\right) m_e l(e) \\
&= \sum_{e\in E(G)} n_e m_e l(e) + \sum_{e\in E(G)} n_e' m_e l(e) \\
&= [\alpha_1, \alpha_2] + \left[\alpha_1', \alpha_2\right].
\end{aligned}$$

Linearity in the second argument follows from linearity in the first argument in combination with the symmetry. ☐

Lemma 2.3.17. *The edge length pairing is retained under refinement.*

Proof. Let G_1, G_2 be two connected, antisymmetric, weighted graphs such that G_2 refines

G_1 and $\alpha_1 = \sum_{e\in E(G_1)} n_e e,\ \alpha_2 = \sum_{e\in E(G_1)} m_e e \in C_1(G_1, A)$. Then the refinement of

$$\begin{aligned}
[\alpha_1, \alpha_2] &= \sum_{e\in E(G_1)} n_e m_e l(e) \\
&= \sum_{e\in E(G_1)} n_e m_e \left(\sum_{e'\in b^{-1}(e)} l(e') \right) \\
&= \sum_{e\in E(G_1)} \sum_{e'\in b^{-1}(e)} n_{b(e')} m_{b(e')} l(e') \\
&= \sum_{e'\in E(G_2)} n_{b(e')} m_{b(e')} l(e') \qquad \text{as } b \text{ is surjective} \\
&= \left[\sum_{e'\in E(G_2)} n_{b(e')} e', \sum_{e'\in E(G_2)} m_{b(e')} e' \right] \\
&= [\alpha_{1,\text{refine}}, \alpha_{2,\text{refine}}]
\end{aligned}$$

□

Definition 2.3.18. A **path** from a vertex V_1 to a vertex V_2 on a graph G is defined to be an element $\alpha = \sum_{e\in E(G)} n_e e \in C_1(G, \mathbb{Z})$ such that

$$\mathrm{d}^*\alpha(V) = \begin{cases} -1 & \text{for } V = V_1 \\ 1 & \text{for } V = V_2 \\ 0 & \text{for all } V \neq V_1, V_2 \end{cases}$$

if $V_1 \neq V_2$ and

$$\mathrm{d}^*\alpha(V) = 0 \text{ for all } V \in V(G)$$

if $V_1 = V_2$.

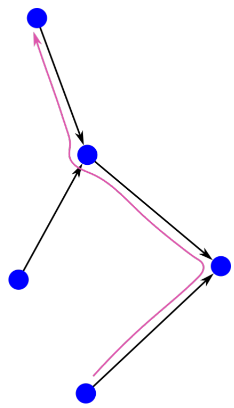


Figure 2.12: A path

Lemma 2.3.19. *Let G be a connected graph and $V_1, V_2 \in V(G)$. Then there exists a path from V_1 to V_2.*

Proof. For $V_1 = V_2$ there exists the trivial path $\alpha = 0$. Assume now $V_1 \neq V_2$. As G is connected by definition, there exists a finite number of vertices $U_0 = V_1, U_1, ..., U_{n-1}, U_n = V_2$ and a finite number of edges $e_1, ..., e_n \in E(G)$ such that $\iota_{\text{ass}}(e_i) = (U_{i-1}, U_i)$. Define now

$$\alpha := \sum_{i=1}^{n} e_i \in C_1(G, \mathbb{Z}).$$

For $i = 1, ..., n-1$ each vertex U_i appears once as tail vertex and once as head vertex of the edges $e_1, ..., e_n \in E(G)$. Hence $\mathrm{d}^*\alpha(U_i) = 0$ for $i = 1, ..., n-1$. As $U_0 = V_1$ appears only as tail vertex, $U_n = V_2$ only as head vertex and all the other vertices do not appear at all, we have

$$\mathrm{d}^*\alpha(V) = \begin{cases} -1 & \text{for } V = V_1 \\ 1 & \text{for } V = V_2 \\ 0 & \text{for all } V \neq V_1, V_2. \end{cases}$$

□

2.4 Raynaud uniformization

Let A be an abelian $\mathbb{C}_p$-variety. Denote the valuation ring of $\mathbb{C}_p$ by $\mathbb{C}_p$ and its maximal ideal by $\mathbb{C}_p$. Its fraction field is then $\mathbb{C}_p/\mathbb{C}_p = \overline{\mathbb{F}}_p = \bigcup_{n\in\mathbb{N}\setminus\{0\}} \mathbb{F}_{p^n}$. The following is a summary of the sections 3.4.1 in [KRZ16b] and 4 in [BR14].

Construction 2.4.1. By Theorem 1.1, [BL91], the $\mathbb{C}_p$-model $\mathcal{A}$ of the abelian $\mathbb{C}_p$-variety A possesses the property that its special fibre $\bar{A} := \mathcal{A}_s$ (s denotes the special fibre) fits into the short exact sequence

$$0 \longrightarrow \bar{T} \longrightarrow \bar{A} \longrightarrow \bar{B} \longrightarrow 0.$$

Denote by M the character lattice of a torus T, then $T = \operatorname{Spec}(\mathbb{C}_p[M])$ and $\bar{T} = \operatorname{Spec}(\overline{\mathbb{F}}_p[M])$. Furthermore $\bar{B}$ is the reduction of an abelian variety B of good reduction. Raynaud showed that there are suitable T and B such that the exact sequence exists.
Now let $\hat{\mathcal{A}}$ be the p-adic completion of $\mathcal{A}$. This is now a so called formal scheme. For more details on formal schemes it is referred to chapter II.9 in [Har77]. $A_0 := (\hat{\mathcal{A}})_\eta$ is defined to be the generic fibre of $\hat{\mathcal{A}}$. This is an analytic subdomain of A^{an} and a formal $\mathbb{C}_p$-Lie subgroup. Since formal Lie groups just play a minor part in this paper, it is referred to chapter 6.5 in [Ber90] for more details on the last facts.
The isomorphism $(\hat{\mathcal{A}})_s \cong \mathcal{A}_s$ delivers

$$\bar{A} = \mathcal{A}_s \cong (\hat{\mathcal{A}})_s =: \bar{A}_0,$$

which transforms the above sequence into

$$0 \longrightarrow \bar{T} \longrightarrow \bar{A}_0 \longrightarrow \bar{B} \longrightarrow 0.$$

Lifting this short exact sequence delivers

$$0 \longrightarrow T_0 \longrightarrow A_0 \longrightarrow B^{\mathrm{an}} \longrightarrow 0.$$

For details see section 4.1, [BR14].

Construction 2.4.2. Let $\pi : \Upsilon \longrightarrow A^{\mathrm{an}}$ be the topological universal cover of A^{an}. Then, after choosing a point in the fibre over $0 \in A^{\mathrm{an}}$ to be the origin, Υ has a unique structure of a $\mathbb{C}_p$-Lie group, such that π is a homomorphism of $\mathbb{C}_p$-Lie groups. Furthermore Υ is the analytification of an algebraic group E, therefore we can write $\Upsilon = E^{\mathrm{an}}$. Since π is a local isomorphism, its kernel

$$M' := \ker(\pi) \cong H_1(A^{\mathrm{an}}, \mathbb{Z})$$

is a discrete subgroup of $E^{\mathrm{an}}(\mathbb{C}_p)$. This delivers the short exact sequence

$$0 \longrightarrow M' \longrightarrow E^{\mathrm{an}} \xrightarrow{\pi} A^{\mathrm{an}} \longrightarrow 0.$$

Furthermore one gets the short exact sequence (4.1.2) from [BR14]

$$0 \longrightarrow T \longrightarrow E \longrightarrow B \longrightarrow 0$$

whose analytification is

$$0 \longrightarrow T^{\mathrm{an}} \longrightarrow E^{\mathrm{an}} \longrightarrow B^{\mathrm{an}} \longrightarrow 0.$$

These results are often summarized in the **Raynaud uniformization cross**:

$$\begin{tikzcd}
 & T^{\mathrm{an}} \arrow[d] & \\
M' \arrow[r] & E^{\mathrm{an}} \arrow[r, "\pi"] \arrow[d] & A^{\mathrm{an}} \\
 & B^{\mathrm{an}} &
\end{tikzcd}$$

Definition 2.4.3. Let M be the character lattice of T, $N = \mathrm{Hom}(M, \mathbb{Z})$ its dual and $\chi^u \in \mathbb{C}_p[M]$ the character of T corresponding to $u \in M$. Then the **tropicalization map** is defined as

$$\begin{aligned} \mathrm{trop} : T^{\mathrm{an}} &\longrightarrow N_{\mathbb{R}} = \mathrm{Hom}(M, \mathbb{R}), \\ ||.|| &\longmapsto \xi_{||.||} \end{aligned}$$

with

$$\begin{aligned} \xi_{||.||} : M &\longrightarrow \mathbb{R}, \\ u &\longmapsto -\log ||\chi^u||. \end{aligned}$$

Construction 2.4.4. The goal is to extend this tropicalization map to the universal cover of A^{an}. Constructions 2.4.1 and 2.4.2 yield the following diagram, consisting of two short exact sequences:

$$\begin{tikzcd} 0 \arrow[r] & T_0 \arrow[r, "\alpha"] \arrow[d, "\iota_1"] & A_0 \arrow[r] \arrow[d, "\iota_2"] & B^{\mathrm{an}} \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & T^{\mathrm{an}} \arrow[r, "\beta"] & E^{\mathrm{an}} \arrow[r] & B^{\mathrm{an}} \arrow[r] & 0 \end{tikzcd}$$

This diagram is commutative as ι_1 and ι_2 are just inclusions. In the case of rigid analysis, Siegfried Bosch and Werner Lütkebohmert showed in section 3 on p. 665 of [BL91] that the rigid analogue of E^{an} is the push-out of the rigid analogues of T^{an} and A_0 with respect to ι_1 and α. Walter Gubler adapted this result to Berkovich theory. In section 4.1, [Gub10] he showed that E^{an} is the push-out of T^{an} and A_0 with respect to ι_1 and α in the category of $\mathbb{C}_p$-analytic groups. For the definition of a K-analytic group and more information about it, it is referred to the beginning of section 5.1, [Ber90].
These results are summarized in the following push-out diagram:

$$\begin{tikzcd} T_0 \arrow[r, "\alpha"] \arrow[d, "\iota_1"] & A_0 \arrow[d, "\iota_2"] \arrow[ddr, bend left, "0"] & \\ T^{\mathrm{an}} \arrow[r, "\beta"] \arrow[drr, bend right, "\mathrm{trop}"] & E^{\mathrm{an}} \arrow[dr, dashed, "\exists!"] & \\ & & N_{\mathbb{R}} \end{tikzcd}$$

It holds $\mathrm{trop} \circ \iota_1 = 0$, as $T_0 = \mathrm{trop}^{-1}(0)$, and therefore $\mathrm{trop} \circ \iota_1 = \alpha \circ 0$. The universal property of the push-out yields now a unique morphism $E^{\mathrm{an}} \longrightarrow N_{\mathbb{R}}$ which will also be denoted by trop.

Corollary 2.4.5. *One has the short exact sequence*

$$0 \longrightarrow A_0 \longrightarrow E^{\mathrm{an}} \xrightarrow{\mathrm{trop}} N_{\mathbb{R}} \longrightarrow 0.$$

Proof. The push-out square in Construction 2.4.4 delivers the sequence

$$0 \longrightarrow A_0 \xrightarrow{\iota_2} E^{\mathrm{an}} \xrightarrow{\mathrm{trop}} N_{\mathbb{R}} \longrightarrow 0.$$

It is exact at

A_0: ι_2 is an inclusion, hence injective.

E^{an}: $\mathrm{Im}(\iota_2) = A_0$ as it is an inclusion. From the first diagram in Construction 2.4.4 it is known that $E^{\mathrm{an}} = \beta(T^{\mathrm{an}}) + \iota_2(A_0)$, hence

$$\begin{aligned} \mathrm{trop}_{E^{\mathrm{an}}}(E^{\mathrm{an}}) &= \mathrm{trop}_{E^{\mathrm{an}}}(\beta(T^{\mathrm{an}}) + \iota_2(A_0)) \\ &= (\mathrm{trop}_{E^{\mathrm{an}}} \circ \beta)(T^{\mathrm{an}}) + (\mathrm{trop}_{E^{\mathrm{an}}} \circ \iota_2)(A_0) \\ &= \mathrm{trop}_{T^{\mathrm{an}}}(T^{\mathrm{an}}) + 0(A_0) \\ &= \mathrm{trop}_{T^{\mathrm{an}}}(T^{\mathrm{an}}) \end{aligned}$$

where $\mathrm{trop}_{E^{\mathrm{an}}}$ respectively $\mathrm{trop}_{T^{\mathrm{an}}}$ denotes the tropicalization map from E^{an} respectively T^{an} to $N_{\mathbb{R}}$. Consequently an element $e \in E^{\mathrm{an}}$ is mapped by $\mathrm{trop}_{E^{\mathrm{an}}}$ to zero if it can be written as a sum

$$e = t_0 + a_0$$

with $t_0 \in T_0 = \mathrm{trop}_{T^{\mathrm{an}}}^{-1}(0)$ and $a_0 \in A_0$. Since T_0 injects into A_0, we have

$$\ker(\mathrm{trop}_{E^{\mathrm{an}}}) = A_0$$

and the sequence is exact at E^{an}.

$N_{\mathbb{R}}$: The map $\mathrm{trop}_{T^{\mathrm{an}}} : T^{\mathrm{an}} \longrightarrow N_{\mathbb{R}}$ is surjective and therefore $\mathrm{trop}_{E^{\mathrm{an}}}$ as well because $\mathrm{trop}_{T^{\mathrm{an}}} = \mathrm{trop}_{E^{\mathrm{an}}} \circ \beta$. This means the sequence is exact at $N_{\mathbb{R}}$. □

Corollary 2.4.6. *Restricting to $\mathbb{C}_p$-rational points delivers the short exact sequence*

$$0 \longrightarrow A_0(\mathbb{C}_p) \longrightarrow E(\mathbb{C}_p) \xrightarrow{\mathrm{trop}} N_{\mathbb{Q}} \longrightarrow 0.$$

Proof. Restricting to $\mathbb{C}_p$-rational points yields the map

$$\mathrm{trop} : E^{\mathrm{an}}(\mathbb{C}_p) \longrightarrow N_{\mathbb{Q}},$$

as $\mathrm{val}(\mathbb{C}_p^\times) = \mathbb{Q}$. On top of that all the $\mathbb{C}_p$-rational points of E^{an} are already in E, which means that $E^{\mathrm{an}}(\mathbb{C}_p) = E(\mathbb{C}_p)$. Finally

$$\ker(\mathrm{trop} : E(\mathbb{C}_p) \longrightarrow N_{\mathbb{Q}}) = A_0 \cap E(\mathbb{C}_p) = A_0(\mathbb{C}_p)$$

as A_0 is a subset of E^{an}. This proves the exactness of the above sequence. □

Example 2.4.7. Corollary 2.4.6 can also be proved without using the reference [Gub10] of Walter Gubler. For this purpose we will restrict to the $\mathbb{C}_p$-rational points of the two short exact sequences in Construction 2.4.4:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T_0(\mathbb{C}_p) & \xrightarrow{\alpha} & A_0(\mathbb{C}_p) & \xrightarrow{\tau_1} & B(\mathbb{C}_p) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \iota_1} & & \downarrow{\scriptstyle \iota_2} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & T(\mathbb{C}_p) & \xrightarrow{\beta} & E(\mathbb{C}_p) & \xrightarrow{\tau_2} & B(\mathbb{C}_p) & \longrightarrow & 0
\end{array}$$

This is a commutative diagram in the category of abelian groups and it is possible to prove the universal property of a push-out by hand for $E(\mathbb{C}_p)$.
The first goal is to show that $E(\mathbb{C}_p)$ is the sum of $\iota_2(A_0(\mathbb{C}_p))$ and $\beta(T(\mathbb{C}_p))$. Let $x \in E(\mathbb{C}_p)$. Then $\tau_2(x) \in B(\mathbb{C}_p)$ and, since τ_1 is surjective, there exists an element $a_0 \in A_0(\mathbb{C}_p)$ such that

$$\tau_1(a_0) = \tau_2(x).$$

Commutativity of the right square yields

$$\tau_1(a_0) = \tau_2(\iota_2(a_0))$$

and hence

$$\tau_2(x - \iota_2(a_0)) = \tau_2(x) - \tau_2(\iota_2(a_0)) = 0,$$

which means $x - \iota_2(a_0) \in \ker(\tau_2) = \mathrm{Im}(\beta)$. Ergo, there exists an element $t \in T(\mathbb{C}_p)$ such that $\beta(t) = x - \iota_2(a_0)$. Written as

$$x = \beta(t) + \iota_2(a_0),$$

this shows the claim.
Commutativity of the left square delivers

$$\beta \circ \iota_1 = \iota_2 \circ \alpha. \tag{2.2}$$

Take now an abelian group G with homomorphisms

$$\beta' : T(\mathbb{C}_p) \longrightarrow G$$

and

$$\iota_2' : A_0(\mathbb{C}_p) \longrightarrow G$$

such that

$$\beta' \circ \iota_1 = \iota_2' \circ \alpha. \tag{2.3}$$

Define a map

$$\begin{aligned} \varphi : E(\mathbb{C}_p) &\longrightarrow G, \\ x = \beta(t) + \iota_2(a_0) &\longmapsto \beta'(t) + \iota_2'(a_0). \end{aligned}$$

To show well-definedness, it suffices to prove $\varphi(0) = 0$.
Let $\beta(t) + \iota_2(a_0) = 0$, meaning

$$\begin{aligned} 0 &= \tau_2(\beta(t) + \iota_2(a_0)) \\ &= \tau_2(\beta(t)) + \tau_2(\iota_2(a_0)) \\ &= 0 + \tau_2(\iota_2(a_0)) \\ &= \tau_1(a_0) \end{aligned}$$

and $a_0 \in \ker(\tau_1) = \mathrm{Im}(\alpha)$. Hence there exists a $t_0 \in T_0(\mathbb{C}_p)$ with $\alpha(t_0) = a_0$.
This and (2.2) yield

$$\begin{aligned} 0 &= \beta(t) + \iota_2(a_0) \\ &= \beta(t) + \iota_2(\alpha(t_0)) \\ &= \beta(t) + \beta(\iota_1(t_0)) \\ &= \beta(t + \iota_1(t_0)) \end{aligned}$$

which delivers $t+\iota_1(t_0) = 0$ in combination with the injectivity of β. This can be written as $t = -\iota_1(t_0)$.
These results give

$$\begin{aligned}\beta'(t) + \iota_2'(a_0) &= \beta'(-\iota_1(t_0)) + \iota_2'(\alpha(t_0))\\ &= \iota_2'(\alpha(t_0)) - \beta'(\iota_1(t_0))\\ &= \left(\iota_2' \circ \alpha - \beta' \circ \iota_1\right)(t_0)\\ &= 0\end{aligned}$$

because of (2.3). This means φ is well-defined. Furthermore it follows directly from the definition that φ is a homomorphism.
The commutativity of the diagram

$$\begin{tikzcd} T_0(\mathbb{C}_p) \arrow[r, "\alpha"] \arrow[d, "\iota_1"] & A_0(\mathbb{C}_p) \arrow[d, "\iota_2"] \arrow[ddr, bend left, "\iota_2'"] & \\ T(\mathbb{C}_p) \arrow[r, "\beta"] \arrow[drr, bend right, "\beta'"'] & E(\mathbb{C}_p) \arrow[dr, dashed, "\varphi"] & \\ & & G \end{tikzcd}$$

follows easily by setting $a_0 = 0$ respectively $t = 0$ in the definition of φ.
It remains to show the uniqueness of φ. Suppose ψ also fulfils the universal property of the push-out, meaning $\psi \circ \beta = \beta'$ and $\psi \circ \iota_2 = \iota_2'$. Then

$$\begin{aligned}\psi(\beta(t) + \iota_2(a_0)) &= (\psi \circ \beta)(t) + (\psi \circ \iota_2)(a_0)\\ &= \beta'(t) + \iota_2'(a_0)\\ &= (\varphi \circ \beta)(t) + (\varphi \circ \iota_2)(a_0)\\ &= \varphi(\beta(t) + \iota_2(a_0))\end{aligned}$$

and therefore $\psi = \varphi$.
It has been shown that, for any abelian group G and homomorphisms β' and ι_2', there exists a unique homomorphism φ, fulfilling the desired properties. This means that the left square of the two short exact sequences on top is a push-out square.
Defining $G := N_{\mathbb{Q}}$, $\beta' :=$ trop and $\iota_2' := 0$ yields the short sequence

$$0 \longrightarrow A_0(\mathbb{C}_p) \longrightarrow E(\mathbb{C}_p) \xrightarrow{\text{trop}} N_{\mathbb{Q}} \longrightarrow 0.$$

Exactness is shown the same way as in the proof of Corollary 2.4.5.

2.5 Differential one-forms

Let R be a ring, A an R-algebra and M an A-module.

Definition 2.5.1. An R-linear **derivation** on A is an R-module homomorphism

$$\mathrm{d}\colon A \longrightarrow M$$

satisfying the Leibniz rule

$$\mathrm{d}(fg) = f\,\mathrm{d}g + g\,\mathrm{d}f.$$

Lemma 2.5.2. *It holds* $\mathrm{d}r = 0$ *for* $r \in R$.

Proof. From $\mathrm{d}r = r\,\mathrm{d}1$ (linearity) and $\mathrm{d}r = \mathrm{d}(1\cdot r) = 1\,\mathrm{d}r + r\,\mathrm{d}1$ (Leibniz rule) it follows

$$\mathrm{d}r = 1\,\mathrm{d}r = 0.$$

□

Definition 2.5.3. $\mathrm{Der}_R(A,M)$ is defined to be the set of R-linear derivations

$$\mathrm{d}\colon A \longrightarrow M.$$

Definition 2.5.4. The module of **Kähler differentials** is defined as the A-module $\Omega_{A/R}$, for which there exists a universal R-linear derivation

$$\mathrm{d}\colon A \longrightarrow \Omega_{A/R}.$$

This means that, if $\mathrm{d}'\colon A \longrightarrow M$ is another R-linear derivation, there exists a unique A-module homomorphism $\varphi : \Omega_{A/R} \longrightarrow M$ such that

$$\begin{tikzcd} A \arrow[rr, "\mathrm{d}"] \arrow[dr, "\mathrm{d}'"'] & & \Omega_{A/R} \arrow[dl, dashed, "\exists!\varphi"] \\ & M & \end{tikzcd}$$

commutes.

Lemma 2.5.5. *This definition yields the natural isomorphism*

$$\mathrm{Der}_R(A,M) \xrightarrow{\sim} \mathrm{Hom}_A(\Omega_{A/R}, M)$$
$$\mathrm{d}' \longmapsto \varphi.$$

Proof. The homomorphism property is clear. Injectivity follows from the uniqueness of φ. Eventually the map is surjective, as for any A-module homomorphism

$$\varphi' : \Omega_{A/R} \longrightarrow M,$$

the composition $\varphi' \circ \mathrm{d}$ is an R-linear derivation $\mathrm{d}\colon A \longrightarrow M$. □

Lemma 2.5.6. *If A is generated by a set $S \subseteq A$ as an R-algebra, then $\Omega_{A/R}$ is generated by $\mathrm{d}S = \{\mathrm{d}s | s \in S\}$ as an A-module.*

Proof. This follows directly from the R-linearity of d and the universal property. □

Example 2.5.7. Let $A = R[x_1, ..., x_n]$, then $\Omega_{A/R}$ is generated by $\mathrm{d}x_1$, ..., $\mathrm{d}x_n$ as an $R[x_1, ..., x_n]$-module, meaning that the elements of $\Omega_{A/R}$ look like

$$f_1 \,\mathrm{d}x_1 + ... + f_n \,\mathrm{d}x_n$$

with $f_1, ..., f_n \in R[x_1, ..., x_n]$.

Example 2.5.8. Let $A \subseteq K[[T, T^{-1}]]$ be the subset of all formal Laurent series that converge on the open annulus around 0, of inner radius r and outer radius R with $r, R \in \mathbb{R}_{>0}$ and $r < R$, then $\Omega_{A/K}$ is generated by $\mathrm{d}T$ and $\mathrm{d}T^{-1}$ as an $K[[T, T^{-1}]]$-module. The K-algebra $A \subseteq K[[T, T^{-1}]]$ is not generated by T and T^{-1} as we do not just have polynomials, but the proof of Lemma 2.5.6 works for Laurent series as well, because they are K-linear. The calculation

$$0 = \mathrm{d}1 = \mathrm{d}(TT^{-1}) = T \,\mathrm{d}T^{-1} + T^{-1} \,\mathrm{d}T$$

yields

$$\mathrm{d}T^{-1} = -T^{-2} \,\mathrm{d}T.$$

This makes $\mathrm{d}T$ the only generator of $\Omega_{A/K}$. Hence we can write any differential one-form $\omega \in \Omega_{A/K}$ as

$$\omega = f \,\mathrm{d}T$$

where $f \in A \subseteq K[[T, T^{-1}]]$ is a formal Laurent series that converges on the mentioned open annulus around 0.

Example 2.5.9. In the case of Example 2.5.8 with $\mathrm{char}(K) = 0$ it holds

$$T^n \,\mathrm{d}T = \mathrm{d}\left(\tfrac{1}{n+1} T^{n+1}\right)$$

for any $n \in \mathbb{Z} \backslash \{-1\}$.

Proof. Case 1: $n \in \mathbb{N}$
The proof will be done by induction. The claim is trivial for $n = 0$. Assume now that it is true for $n - 1 \in \mathbb{N}$, that means $nT^{n-1} \,\mathrm{d}T = \mathrm{d}T^n$. Then

$$\begin{aligned} \mathrm{d}T^{n+1} &= T^n \,\mathrm{d}T + T \,\mathrm{d}T^n \\ &= T^n \,\mathrm{d}T + TnT^{n-1} \,\mathrm{d}T \\ &= T^n \,\mathrm{d}T + nT^n \,\mathrm{d}T \\ &= (n+1)T^n \,\mathrm{d}T \end{aligned}$$

which delivers the claim, as $n+1$ is invertible in K.

Case 2: $-n \in \mathbb{N}\backslash\{0,1\}$
Let $m := -n$ and try to prove the claim

$$T^{-m}\,\mathrm{d}T = \mathrm{d}\left(\tfrac{1}{-m+1}T^{-m+1}\right)$$

The proof will again be done by induction. For $m=2$ the initial step

$$\mathrm{d}T^{-1} = -T^{-2}\,\mathrm{d}T$$

comes from Example 2.5.8. Assume now that it is true for $m \in \mathbb{N}\backslash\{0,1\}$. The aim is to prove the equation for $m+1$.

$$\begin{aligned}
\mathrm{d}T^{-m} &= T^{-m+1}\,\mathrm{d}T^{-1} + T^{-1}\,\mathrm{d}T^{-m+1}\\
&= T^{-m+1}(-T^{-2}\,\mathrm{d}T) + T^{-1}(-m+1)T^{-m}\,\mathrm{d}T\\
&= -T^{-m-1}\,\mathrm{d}T + (-m+1)T^{-m-1}\,\mathrm{d}T\\
&= -mT^{-m-1}\,\mathrm{d}T
\end{aligned}$$

delivers

$$\mathrm{d}\left(\tfrac{1}{-m}T^{-m}\right) = T^{-(m+1)}\,\mathrm{d}T$$

because $-m$ is invertible, which is the claim for $m+1$. $\square$

Remark 2.5.10. Due to Example 2.5.9, for any formal Laurent series

$$\sum_{n\in\mathbb{Z}\backslash\{-1\}} a_nT^n,$$

the equality

$$\Big(\sum_{n\in\mathbb{Z}\backslash\{-1\}} a_nT^n\Big)\,\mathrm{d}T = \mathrm{d}\Big(\sum_{n\in\mathbb{Z}\backslash\{-1\}} \tfrac{1}{n+1}a_nT^{n+1}\Big)$$

holds. The transformation of the formal Laurent series from above into the formal Laurent series on the right-hand side of the equality is called **formal antidifferentation**.

Definition 2.5.11. Assume that $X = \mathrm{Spec}(A) \longrightarrow Y = \mathrm{Spec}(R)$ is a morphism of affine schemes. The **sheaf of Kähler differentials** $\Omega_{X/Y}$ is the quasi-coherent sheaf associated to the module of Kähler differentials $\Omega_{A/R}$. For a scheme X over a field K it is written $\Omega_{X/K}$ instead of $\Omega_{X/Y}$, for $R=K$ and hence $Y=\mathrm{Spec}(K)$. The elements $\omega \in \Omega_{X/K}$ are also called **differential one-forms** on X. For this reason one may also write $\Omega^1_{X/K}$ instead of $\Omega_{X/K}$.

Remark 2.5.12. In the case that X is a scheme over a field K, one has $A = \mathcal{O}_X$, the structure sheaf of X. Hence the universal derivation is a map

$$\mathrm{d}\colon \mathcal{O}_X \longrightarrow \Omega_{X/K}.$$

This derivation extends now in a natural way to a sequence of maps

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\mathrm{d}} \Omega_{X/K} \xrightarrow{\mathrm{d}} \Omega^2_{X/K} \xrightarrow{\mathrm{d}} \Omega^3_{X/K} \xrightarrow{\mathrm{d}} \dots$$

with $\Omega^n_{X/K} := \bigwedge_n \Omega_{X/K}$. This complex is called the **de Rham complex** and fulfils $\mathrm{d} \circ \mathrm{d} = 0$. Since this construction is not important for this paper, it is referred to literature about de Rham cohomology for further information.

Definition 2.5.13. A differential one-form $\omega \in \Omega_{X/K}$ is called **closed** if $\mathrm{d}\omega = 0$. The set of closed differential one-forms in $\Omega_{X/K}$ is denoted by $Z^1_{\mathrm{dR}}(X)$.

Definition 2.5.14. A differential one-form $\omega \in \Omega_{X/K}$ is called **exact** if there exists a function $F \in \mathcal{O}_X$ such that $\omega = \mathrm{d}F$.

Lemma 2.5.15. *Every exact differential one-form is closed.*

Proof. $\mathrm{d}\omega = \mathrm{dd}F = 0$ as $\mathrm{d} \circ \mathrm{d} = 0$ holds in the de Rham complex. □

Definition 2.5.16. Assume that X has additionally the structure of a K-Lie group, where K is a non-archimedean field. A differential one-form $\omega \in \Omega_{X/K}$ is called **invariant** if

$$(L_x)^*\omega = \omega$$

and

$$(R_x)^*\omega = \omega$$

where L_x respectively R_x are the translations on X by left respectively right addition of x. The set of invariant differential one-forms in $\Omega_{X/K}$ is denoted by $\Omega^1_{\mathrm{inv}}(X)$.

2.6 Skeletons

Let K be a field that is algebraically closed and complete with respect to a nontrivial, non-archimedean valuation $\mathrm{val} : K \longrightarrow \mathbb{R} \cup \{\infty\}$. The valuation ring of K will be denoted by R, its maximal ideal by $\mathfrak{m}$ and its fraction field $R/\mathfrak{m}$ by k. This section will be restricted to the case $K = \mathbb{C}_p$.

Definition 2.6.1. Let X be a smooth, proper, connected $\mathbb{C}_p$-curve. A proper, connected, flat R-scheme $\mathcal{X}$ of relative dimension 1 is called a **semistable model** of X if its generic fibre is equal to the smooth curve X and the only singularities of its special fibre $\mathcal{X}_k$ are double points.

Remark 2.6.2. This semistable model creates a reduction map between X^{an} and $\mathcal{X}_k$. The whole situation is depicted in the following figure.

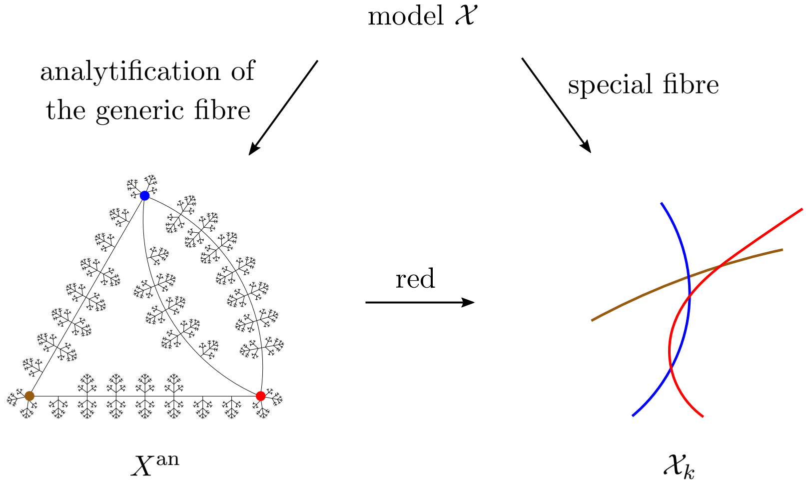


Figure 2.13: Interplay of the model $\mathcal{X}$ and the curve X

Theorem 2.6.3. *Let $\mathcal{X}$ be a semistable model, $X = \mathcal{X}_K$ its generic fibre and $\tilde{x} \in \mathcal{X}_k$ a point in the special fibre. Then*

(i) $\tilde{x}$ is a generic point if and only if $\mathrm{red}^{-1}(\tilde{x})$ is a singleton (called vertex),
(ii) $\tilde{x}$ is a smooth point if and only if $\mathrm{red}^{-1}(\tilde{x}) \cong B(1)_+$ and
(iii) $\tilde{x}$ is a double point if and only if $\mathrm{red}^{-1}(\tilde{x}) \cong S(\rho)_+$ for some $\rho \in |\mathbb{C}_p^\times|$.

Proof. Theorem 3.2.4, [KRZ16a]. □

Construction 2.6.4. Consider the tropicalization map from Definition 2.4.3 and choose the character lattice M to be

$$\{..., T^{-2}, T^{-1}, 1, T, T^2, ...\}$$

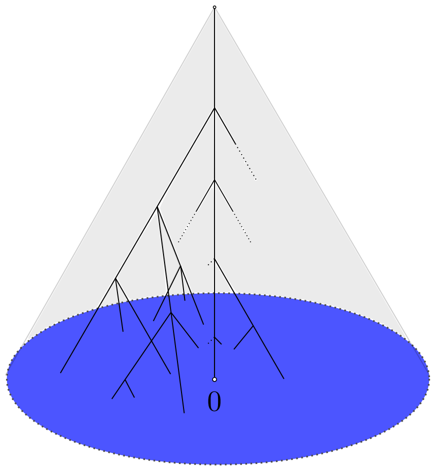
where this T has to be considered as a symbol and not to be mistaken with the torus T. This yields the torus

$$T = \mathrm{Spec}(\mathbb{C}_p[T, T^{-1}]).$$

Its analytification is the set of all multiplicative seminorms on $\mathbb{C}_p[T, T^{-1}]$ that continues the absolute value on $\mathbb{C}_p$. This is a Berkovich analytic space with net

$$\tau = \{S(R, 0)\},$$

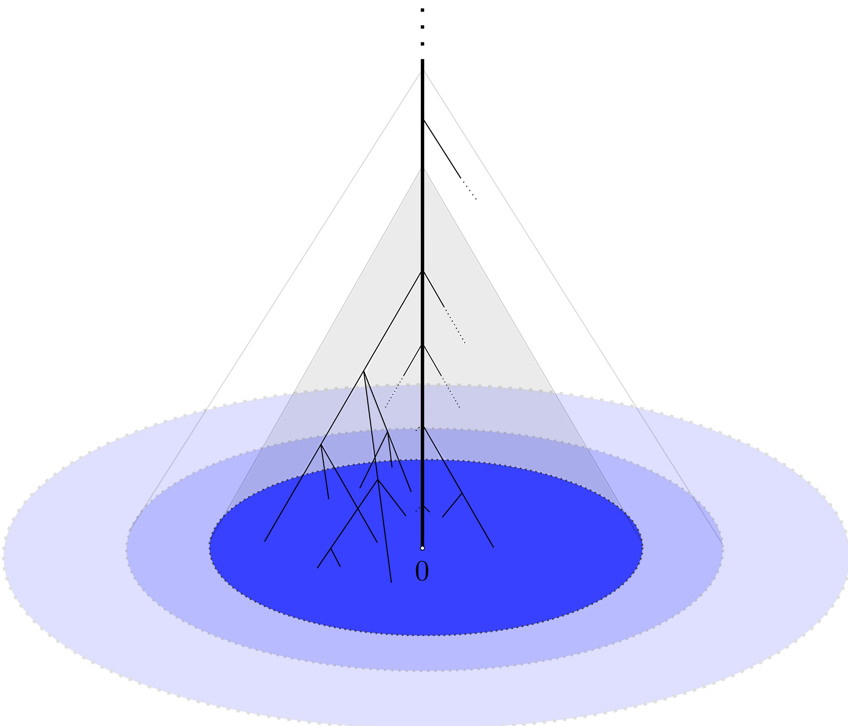
where $R \in \mathbb{R}$ and $S(R, 0)$ is the set of all multiplicative seminorms on the Laurent series that converges for all $T \in \mathbb{C}_p$ with $0 < |T| < R$. In the case $R = 1$, one gets $S(1, 0) = S(0)_+$.



$\{x \in \mathbb{C}_p | 0 < |x|_p < R\}$

Figure 2.14: $S(R, 0)$

Taking the limit $R \longrightarrow +\infty$, the analytification T^{an} of the torus may be depicted as



$\{x \in \mathbb{C}_p | 0 < |x|_p\}$

Figure 2.15: T^{an}

where the bold line in the middle represents all multiplicative seminorms

$$\begin{aligned} \zeta_{a,r} : \mathbb{C}_p[T, T^{-1}] &\longrightarrow \mathbb{C}_p, \\ f(T) &\longmapsto \sup_{\substack{|y-a|_p \leq r \\ y \neq 0}} |f(y)|, \end{aligned}$$

with $a = 0$ for $r \in |\mathbb{C}_p^\times|$.
For $M = \{..., T^{-2}, T^{-1}, 1, T, T^2, ...\}$, the tropicalization map is defined on T^{an} by

$$\begin{aligned} \mathrm{trop} : T^{\mathrm{an}} &\longrightarrow N_{\mathbb{R}} = \mathrm{Hom}(M, \mathbb{R}), \\ ||.|| &\longmapsto \xi_{||.||} \end{aligned}$$

with

$$\begin{aligned} \xi_{||.||} : M &\longrightarrow \mathbb{R}, \\ T^n &\longmapsto -\log ||T^n|| \end{aligned}$$

Choosing the basis of M to be the symbol T, one can identify the homomorphism $\xi_{||.||}$ with the image of the basis element T, meaning that $-\log ||T|| \in \mathbb{R}$. This leads to the map

$$\begin{aligned} \mathrm{trop} : T^{\mathrm{an}} &\longrightarrow \mathbb{R}, \\ ||.|| &\longmapsto -\log ||T|| \end{aligned}$$

and, if the notation $\zeta_{a,r}$ is used for the multiplicative seminorm $||.||$, this becomes

$$\begin{aligned} \mathrm{trop} : T^{\mathrm{an}} &\longrightarrow \mathbb{R}, \\ \zeta_{a,r} &\longmapsto -\log \Big(\sup_{\substack{|y-a|_p \leq r \\ y \neq 0}} |y| \Big). \end{aligned} \tag{2.4}$$

Since

$$|y|_p \leq \max(|y-a|_p, |a|_p)$$

and

$$|a-a|_p = 0 \leq r$$

hold, the equation

$$\sup_{\substack{|y-a|_p \leq r \\ y \neq 0}} |y|_p = |a|_p$$

is true in the case $a \neq 0$. For $a = 0$ it holds

$$\sup_{\substack{|y-0|_p \leq r \\ y \neq 0}} |y| = r.$$

Hence (2.4) transforms into

$$\begin{aligned} \mathrm{trop} : T^{\mathrm{an}} &\longrightarrow \mathbb{R}, \\ \zeta_{a,r} &\longmapsto \begin{cases} -\log(|a|_p) & \text{if } a \neq 0 \\ -\log(r) & \text{if } a = 0. \end{cases} \end{aligned}$$

Consider a point $\zeta_{a,r}$ for $a \neq 0$ in Figure 2.15. Going upwards along the branches, one arrives sometime at the bold line in the middle. The first point that one reaches there is the element ζ_{0,r_0} with the following property: a must lie in the domain of ζ_{0,r_0}, meaning $|a|_p \leq r_0$ and for any $r_1 < r_0$ it must not lie in the domain of ζ_{0,r_1}, which means $|a|_p > r_1$. Consequently one gets

$$r_0 = |a|_p.$$

This process makes it possible to map every $\zeta_{a,r}$ with $a \neq 0$ to $\zeta_{0,|a|_p}$, and to hold every $\zeta_{a,r}$ with $a = 0$.

$$\zeta_{0,r} \longmapsto r$$

delivers an identification of the bold line in Figure 2.15 with $|\mathbb{C}_p^\times|$. Under this identification, the process may be described by the map

$$\begin{aligned} \mathfrak{abs} : T^{\mathrm{an}} &\longrightarrow |\mathbb{C}_p^\times|, \\ \zeta_{a,r} &\longmapsto \begin{cases} |a|_p & \text{if } a \neq 0 \\ r & \text{if } a = 0. \end{cases} \end{aligned}$$

$-\log$ maps finally $|\mathbb{C}_p^\times|$ into $\mathbb{R}$, and this composition gives the tropicalization map

$$\mathrm{trop} = -\log \circ \mathfrak{abs}\,.$$

The process created a descriptive imagination of the tropicalization map. It takes points in the analytification T^{an} back to the bold line in the middle. Let ζ_{0,r_0} be the first point that is reached there, then one finally gets $-\log(r_0)$.
One can restrict this process to $S(\rho)_+$, which may be considered as a subset of T^{an}. But then, only the points ζ_{0,r_0} on the bold line, with $\rho < r_0 < 1$, will be reached. These points build the **skeleton** $\Sigma\,(S(\rho)_+)$ of the open annulus $S(\rho)_+$, and the process is called **retraction**. Eventually the retraction reaches just values between $-\log(1) = 0$ and $-\log(\rho)$. This is an interval on $\mathbb{R}$ with length $-\log(\rho)$. This is defined to be the associated length of $\Sigma\,(S(\rho)_+)$. The retraction on $S(\rho)_+$ may be identified with the tropicalization map. Furthermore the retraction can also be applied to the open annuli in the semistable decomposition as they are isomorphic to $S(\rho)_+$.

Definition 2.6.5. By Corollary 3.2.5, [KRZ16a], the analytification X^{an} is a union of finitely many singletons, infinitely many open balls isomorphic to $B(1)_+$ and finitely many open annuli isomorphic to $S(\rho)_+$, for possibly different $\rho \in |\mathbb{C}_p^\times|$. This is called the **semistable decomposition** of X^{an}.

Definition 2.6.6. The **skeleton** $\Gamma_{\mathcal{X}}$ of X with respect to the semistable model $\mathcal{X}$ is defined to be the union of all singletons and all the subsets of X^{an} that are isomorphic to the skeletons of the open annuli in the semistable decomposition. Note that $\Gamma_{\mathcal{X}}$ is a subset of X^{an}.

Construction 2.6.7. One may associate a weighted graph to the skeleton $\Gamma_{\mathcal{X}}$ in the following way. Take the singletons to be vertices. The annuli are the pre-images of double points, which come from intersections of respectively two irreducible components, whose pre-images are two (possibly equal) singletons. Then one constructs an edge between the vertices belonging to these two singletons. One takes the length of this edge to be its associated length $-\log(\rho)$. Finally an orientation of the edge has to be chosen. This choice is free but it is necessary to always choose the same orientation between two vertices, such that the graph becomes antisymmetric. Since X is connected, $\mathcal{X}_k$ is connected and hence the graph will be connected. By this, one gets a connected, antisymmetric, weighted graph that will be denoted by Γ.

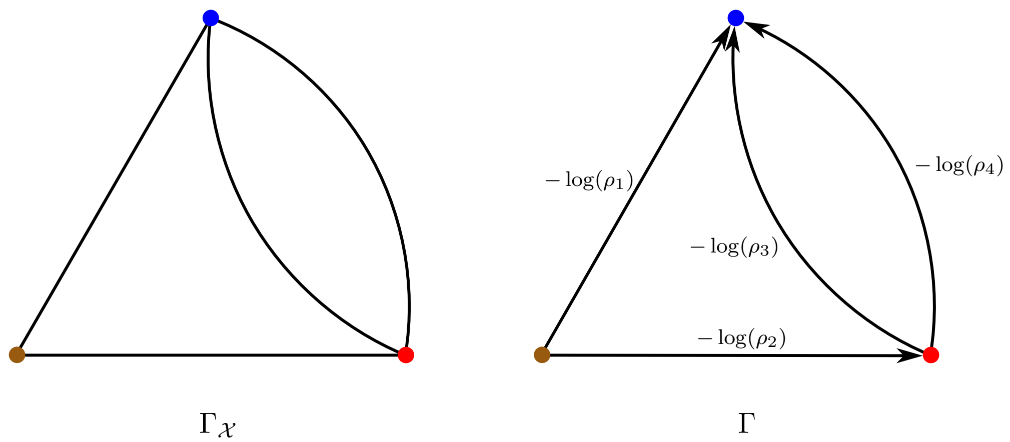


Figure 2.16: A connected, antisymmetric, weighted graph Γ associated to $\Gamma_{\mathcal{X}}$

While it is common to use Γ for both Γ and $\Gamma_{\mathcal{X}}$, in the following $\Gamma_{\mathcal{X}}$ will mean the subset of X^{an} whereas Γ denotes the graph corresponding to the skeleton $\Gamma_{\mathcal{X}}$.

Definition 2.6.8. One defines the retraction map

$$\tau : X^{\mathrm{an}} \longrightarrow \Gamma_{\mathcal{X}}$$

in the following way: If x is a vertex, $\tau(x) := x$ will be fixed. An open ball in the semistable decomposition corresponds to a smooth point in $\mathcal{X}_k$, which lies on an irreducible component, which corresponds to a vertex in X^{an}. Then the open ball is called adjacent to this vertex and any point in this open ball is mapped to this vertex by τ. Finally, on the open annuli, the retraction map from Construction 2.6.4 will be taken. The retractions of the open annuli are called open edges in $\Gamma_{\mathcal{X}}$.

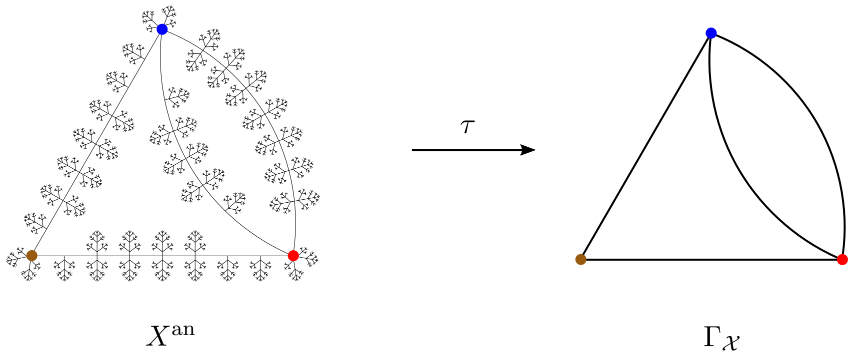


Figure 2.17: Retraction of X^{an}

3 Integration theories

The first step is to define in general what an integration theory in the non-archimedean world should be. For this purpose, the most important properties from the complex integral will be taken and required for p-adic integrals.

3.1 Complex integral

Before starting to define a p-adic integration theory, it is worth to repeat the complex line integral.

Definition 3.1.1. Let X be a topological space. A **path** on X is defined to be a continuous function

$$\gamma : [a, b] \longrightarrow X$$

with real numbers $a \leq b$.

Definition 3.1.2. Let $f : U \longrightarrow \mathbb{C}$ be a continuous function on an open subset $U \subseteq \mathbb{C}$, and $\gamma : [0, 1] \longrightarrow U$ be a path in U. Then

$$\int_\gamma f(z)\ \mathrm{d}z := \int_0^1 f(\gamma(t))\gamma'(t)\ \mathrm{d}t$$

where the latter is calculated by splitting in the real and imaginary part, and applying the real Lebesgue integral.
One defines the complex line integral for general differential one-forms by covering the path with open balls, and applying the Poincaré lemma, which says that every closed differential one-form is exact on an open ball.

Lemma 3.1.3. *Let, in the definition above, U be an open ball and $\omega = \mathrm{d}F$ with F holomorphic on U. Then*

$$\int_\gamma \omega = F(\gamma(1)) - F(\gamma(0)).$$

Proof.

$$\begin{aligned}\int_\gamma \omega = \int_\gamma \mathrm{d}F = \int_\gamma \frac{\mathrm{d}F}{\mathrm{d}z}\ \mathrm{d}z &= \int_0^1 F'\left(\gamma(t)\right)\gamma'(t)\ \mathrm{d}t \\ &= \int_0^1 \left(\frac{\mathrm{d}}{\mathrm{d}t}F(\gamma(t))\right)\ \mathrm{d}t = F(\gamma(1)) - F(\gamma(0))\end{aligned}$$

□

Lemma 3.1.4. *$\int_\gamma \omega$ only depends on the fixed end-point homotopy class of γ.*

Proof. Cauchy's integral theorem □

Remark 3.1.5. It is clear that, if one can concatenate two paths, the integral of the concatenation is equal to the sum of the two integrals of the single paths. Furthermore the line integral is linear in ω, which follows directly from the definition.

3.2 p-adic integration theories

The goal is to maintain these properties of the complex line integral also for the p-adic integral.

Definition 3.2.1. Let X be a smooth $\mathbb{C}_p$-analytic space, and let $\mathcal{P}(X)$ be the set of paths $\gamma : [0,1] \longrightarrow X$ with endings in $X(\mathbb{C}_p)$. An **integration theory** on X is a map

$$\int : \mathcal{P}(X) \times Z^1_{\mathrm{dR}}(X) \longrightarrow \mathbb{C}_p$$

such that:

(i) If $U \subseteq X$ is an open subdomain isomorphic to an open ball, and $\omega = \mathrm{d}f$ with f analytic on U, then

$$\int_\gamma \omega = f(\gamma(1)) - f(\gamma(0))$$

for all $\gamma : [0,1] \longrightarrow U$.

(ii) $\int_\gamma \omega$ only depends on the fixed end point homotopy class of γ.

(iii) If $\gamma_1, \gamma_2 \in \mathcal{P}(X)$ and $\gamma_2(0) = \gamma_1(1)$, then

$$\int_{\gamma_1 * \gamma_2} \omega = \int_{\gamma_1} \omega + \int_{\gamma_2} \omega,$$

where $\gamma_1 * \gamma_2$ is the concatenation of the two paths.

(iv) $\omega \longmapsto \int_\gamma \omega$ is linear in ω for a fixed γ.

Remark 3.2.2. A priori this is just a definition and one does not know whether such an integration theory in the non-archimedean world actually exists. Even if an integration theory exists, it does not have to be unique, as a path in a smooth $\mathbb{C}_p$-analytic space in general cannot be covered by open balls.
Therefore in the following two chapters, two integration theories are going to be introduced. Subsequently, they will be compared in the last chapter.

4 p-adic abelian integral

The start will be done with an approach that uses p-adic Lie theory. It was treated by Yuri G. Zarhin in great generality and extended by Pierre Colmez. Its main feature is that it totally omits using paths.

4.1 p-adic abelian logarithm

Let K be a complete subfield of $\mathbb{C}_p$, and A an abelian variety over K.

Theorem 4.1.1. *There exists a unique homomorphism of K-Lie groups*

$$\log_A : A(K) \longrightarrow \mathrm{Lie}(A)$$

such that $\mathrm{d}\log_A : \mathrm{Lie}(A) \longrightarrow \mathrm{Lie}(A)$ *is the identity.* $\mathrm{Lie}(A)$ *is the short notation of* $\mathrm{Lie}(A(K))$.

In order to prove this theorem, firstly the following theory is needed:

Theorem 4.1.2. *Suppose A is an abelian variety over $\mathbb{C}_p$ and H is an open subgroup in the canonical topology of $A(\mathbb{C}_p)$ where we consider $A(\mathbb{C}_p)$ as a $\mathbb{C}_p$-Lie group and take the topology of the underlying $\mathbb{C}_p$-analytic manifold. Then $A(\mathbb{C}_p)/H$ is a torsion group.*

Proof. Let $x \in A(\mathbb{C}_p)$. The goal is to find an $n \in \mathbb{N}\backslash\{0\}$ such that $nx \in H$.

Step 1: Find an $n_1 \in \mathbb{N}\backslash\{0\}$ such that $n_1 x \in A_0(\mathbb{C}_p)$.

Corollary 2.4.6 delivers the short exact sequence

$$0 \longrightarrow A_0(\mathbb{C}_p) \longrightarrow E(\mathbb{C}_p) \xrightarrow{\mathrm{trop}} N_{\mathbb{Q}} \longrightarrow 0.$$

This gives the isomorphism

$$E(\mathbb{C}_p)/A_0(\mathbb{C}_p) \cong N_{\mathbb{Q}}. \tag{4.1}$$

Furthermore one has the short exact sequence from Construction 2.4.2

$$0 \longrightarrow M' \longrightarrow E^{\mathrm{an}} \xrightarrow{\pi} A^{\mathrm{an}} \longrightarrow 0,$$

where M' is a discrete subgroup of $E^{\mathrm{an}}(\mathbb{C}_p)$. Hence one gets isomorphisms $E^{\mathrm{an}}/M' \cong A^{\mathrm{an}}$ and $E^{\mathrm{an}}(\mathbb{C}_p)/M' \cong A^{\mathrm{an}}(\mathbb{C}_p)$ that can be written as

$$E(\mathbb{C}_p)/M' \cong A(\mathbb{C}_p). \tag{4.2}$$

A_0 injects into A, that means it has trivial intersection with M'. Choose a basis for N and denote its rank by r. Since $\mathrm{trop}(M')$ is a full-rank lattice in $N_{\mathbb{Q}}$, the quotient of (4.1) by M' is

$$\frac{E(\mathbb{C}_p)/M'}{A_0(\mathbb{C}_p)/(M' \cap A_0(\mathbb{C}_p))} \cong (\mathbb{Q}/\mathbb{Z})^r.$$

The left side can be written as

$$\frac{E(\mathbb{C}_p)/M'}{A_0(\mathbb{C}_p)/(M' \cap A_0(\mathbb{C}_p))} = \frac{E(\mathbb{C}_p)/M'}{A_0(\mathbb{C}_p)/0} \overset{(4.2)}{\cong} A(\mathbb{C}_p)/A_0(\mathbb{C}_p)$$

which delivers the isomorphism

$$A(\mathbb{C}_p)/A_0(\mathbb{C}_p) \cong (\mathbb{Q}/\mathbb{Z})^r.$$

Since $(\mathbb{Q}/\mathbb{Z})^r$ is torsion, the same holds for $A(\mathbb{C}_p)/A_0(\mathbb{C}_p)$ and there exists an $n_1 \in \mathbb{N}\backslash\{0\}$ such that $n_1 x \in A_0(\mathbb{C}_p)$.

Step 2: Find an $n_2 \in \mathbb{N}\backslash\{0\}$ such that $n_2(n_1 x) \in A_0(\mathbb{C}_p)$.

The model of A (see Construction 2.4.1 for more details) delivers a reduction map

$$\mathrm{red} : A_0 = (\hat{\mathcal{A}})_\eta \longrightarrow (\hat{\mathcal{A}})_s$$

with $A_0 = (\hat{\mathcal{A}})_\eta$, which may be restricted to $\mathbb{C}_p$-rational points:

$$\mathrm{red} : A_0(\mathbb{C}_p) \longrightarrow (\hat{\mathcal{A}})_s$$

The kernel $\mathrm{red}^{-1}(0)$ of this reduction map is a subgroup of $A_0(\mathbb{C}_p)$ and a $\mathbb{C}_p$-analytic domain that is isomorphic to the open ball of radius 1,

$$\mathrm{red}^{-1}(0) \cong \left\{(z_1, ..., z_r)|z_i \in \mathbb{C}_p\right\}, \tag{4.3}$$

which corresponds to $A_0(\mathbb{C}_p)$.

$$A_0(\overline{\mathbb{F}}_p) = A_0(\mathbb{C}_p/\mathbb{C}_p) = A_0(\mathbb{C}_p)/A_0(\mathbb{C}_p) = A_0(\mathbb{C}_p)/A_0(\mathbb{C}_p)$$

is torsion, as any number $z \in \overline{\mathbb{F}}_p = \bigcup_{n \in \mathbb{N}\backslash\{0\}} \mathbb{F}_{p^n}$ is contained in the finite field $\mathbb{F}_{p^n}$ for a certain $n \in \mathbb{N}\backslash\{0\}$, meaning that z is finite. This means that there exists an $n_2 \in \mathbb{N}\backslash\{0\}$ such that $n_2(n_1 x) \in A_0(\mathbb{C}_p)$.

Step 3: Find an $n_3 \in \mathbb{N}\backslash\{0\}$ such that $n_3(n_2(n_1 x)) \in H$.

$n_2(n_1 x) \in A_0(\mathbb{C}_p)$ can be identified with an element from $\left\{(z_1, ..., z_r)|z_i \in \mathbb{C}_p\right\}$ by (4.3). The aim is to show that the multiplication with the number p is a contraction on $A_0(\mathbb{C}_p)$. Note that A_0 is a formal group, and hence $A_0(\mathbb{C}_p)$ is an r-dimensional formal group over the ring $\mathbb{C}_p$, as $\mathbb{C}_p \subseteq \mathbb{C}_p$. With the definition of an r-dimensional formal group over a ring R in section II.9.1, [Haz78], one gets for the corresponding formal group law F, which always comes along with a formal group, the following structure

$$\begin{aligned} F : A_0(\mathbb{C}_p) \times A_0(\mathbb{C}_p) &\longrightarrow A_0(\mathbb{C}_p) \\ (x, y) &\longmapsto \begin{pmatrix} F_1(x, y) \\ . \\ F_r(x, y) \end{pmatrix} \end{aligned}$$

with

$$x = \begin{pmatrix} x_1 \\ . \\ x_r \end{pmatrix}, y = \begin{pmatrix} y_1 \\ . \\ y_r \end{pmatrix}$$

and $x_i, y_i \in \mathbb{C}_p \subseteq \mathbb{C}_p$ for $i = 1, ..., r$. In this law, $F_i(x, y)$ are power series from $\mathbb{C}_p[[x_i, y_i]]$ and fulfill the properties

(i) $F_i(x, F_i(y, z)) = F_i(F_i(x, y), z)$ and
(ii) $F_i(x, y) = x_i + y_i +$ terms of degree at least 2

for $i = 1, ..., r$.
Adding an element $z \in A_0(\mathbb{C}_p)$ exactly p times gives

$$\begin{aligned} pz &= \underbrace{F(z, F(z, F(...)))}_{p-1 \text{ times}} \\ &= \begin{pmatrix} \underbrace{z_1 + ... + z_1}_{p \text{ times}} + \text{terms of degree at least 2} \\ ... \\ \underbrace{z_r + ... + z_r}_{p \text{ times}} + \text{terms of degree at least 2} \end{pmatrix} \\ &= \begin{pmatrix} pz_1 + \text{ terms of degree at least 2} \\ ... \\ pz_r + \text{ terms of degree at least 2} \end{pmatrix}. \end{aligned}$$

Apply this to $z := n_2(n_1 x) \in \{(z_1, ..., z_r) | z_i \in \mathbb{C}_p\}$. The entries of the vector pz have the form

$$pz_i + \text{terms of degree at least } 2 = pz_i + a_2 z_i^2 + a_3 z_i^3 + ...$$

with $a_i \in \mathbb{C}_p$, that means $|a_i| \leq 1$. Hence

$$\left| pz_i + a_2 z_i^2 + a_3 z_i^3 + ... \right| \leq \max(|pz_i|, |a_2 z_i^2|, |a_3 z_i^3|, ...).$$

$|p| < 1$ and $|z_i| < 1$ deliver

$$\max(|pz_i|, |a_2 z_i^2|, |a_3 z_i^3|, ...) < |z_i|$$

for all $i \in \{1, ..., r\}$. Consequently the norm of pz must be smaller than the norm of z. This means that the multiplication with p is a contraction on $A_0(\mathbb{C}_p)$, which delivers the convergence

$$p^m z \xrightarrow[m \to +\infty]{} 0.$$

Since H is an open subgroup containing 0, there exists an $m_0 \in \mathbb{N}$ such that $p^{m_0} z \in H$. Define $n_3 := p^{m_0} \in \mathbb{N}$. Then it follows $n_3 z = n_3(n_2(n_1 x)) \in H$.

Conclusion:

Finally state $n := n_1 n_2 n_3$ and get $nx \in H$. This means $A(\mathbb{C}_p)/H$ is a torsion group.

$\square$

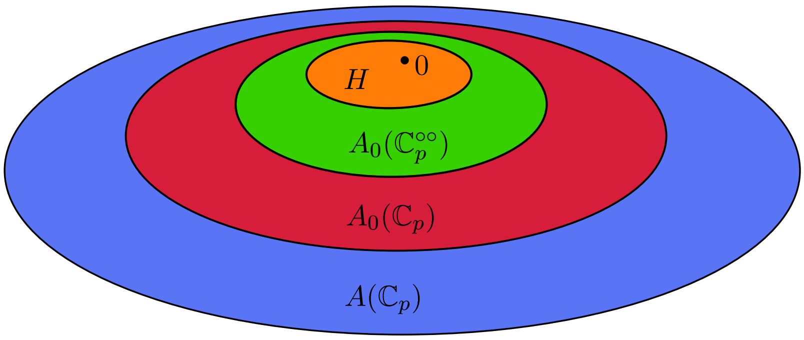


Figure 4.1: Venn diagram of the setting

Example 4.1.3. The proof of Theorem 4.1.2 is quite abstract. For this reason, an example in the one dimensional case is given here. More precisely, A is an elliptic curve with bad reduction, meaning that there exists $q \in \mathbb{C}_p, 0 < |q| < 1$ such that

$$A(\mathbb{C}_p) \cong \mathbb{C}_p^\times / q^{\mathbb{Z}}.$$

This delivers immediately the short exact sequence

$$0 \longrightarrow q^{\mathbb{Z}} \longrightarrow \mathbb{C}_p^\times \xrightarrow{\pi} \mathbb{C}_p^\times / q^{\mathbb{Z}} \longrightarrow 0.$$

Obviously

$$0 \longrightarrow \{x \in \mathbb{C}_p |\, \mathrm{val}(x) = 0\} \longrightarrow \mathbb{C}_p^\times \xrightarrow{\mathrm{val}} \mathbb{Q} \longrightarrow 0$$

is another exact sequence. It holds

$$\mathrm{val}(q^{\mathbb{Z}}) \subseteq \mathrm{val}(\mathbb{C}_p^\times) = \mathbb{Q}$$

and hence

$$\mathbb{Z} \cdot \operatorname{val}(q) \subseteq \mathbb{Q}.$$

<u>Goal</u>: For a given $x \in A(\mathbb{C}_p)$ find an $n \in \mathbb{N}\backslash\{0\}$ such that $x^n \in H$, where H is an open subgroup.

Let $x \in A(\mathbb{C}_p) \cong \mathbb{C}_p^\times / q^{\mathbb{Z}}$. Choose a representative $y \in \mathbb{C}_p^\times$ such that $x = \pi(y)$.

<u>Step 1</u>: Find an $n_1 \in \mathbb{N}\backslash\{0\}$ such that $x^{n_1} = \pi(z)$, with $\operatorname{val}(z) = 0$, $z \in \mathbb{C}_p^\times$.

Since $\operatorname{val}(\mathbb{C}_p^\times) = \mathbb{Q}$, we can write $\operatorname{val}(q) = \frac{a}{b}$ and $\operatorname{val}(y) = \frac{c}{d}$, with $a, b, d \in \mathbb{N}\backslash\{0\}, c \in \mathbb{Z}$. It is correct to assume $a \in \mathbb{N}\backslash\{0\}$ instead of $a \in \mathbb{Z}$ because $0 < |q| < 1$ and thus $\operatorname{val}(q) > 0$. This gives

$$ad \cdot \operatorname{val}(y) = ad \cdot \tfrac{c}{d} = ac = bc \cdot \tfrac{a}{b} = bc \cdot \operatorname{val}(q) \in \mathbb{Z} \cdot \operatorname{val}(q)$$

and hence

$$\operatorname{val}(y^{ad}) = ad \cdot \operatorname{val}(y) \in \mathbb{Z} \cdot \operatorname{val}(q).$$

Consequently there exists an $m \in \mathbb{Z}$ such that

$$\operatorname{val}(y^{ad}) = m \cdot \operatorname{val}(q).$$

It yields

$$\operatorname{val}(y^{ad}) = \operatorname{val}(q^m)$$

and

$$\operatorname{val}\left(\frac{y^{ad}}{q^m}\right) = \operatorname{val}(y^{ad}) - \operatorname{val}(q^m) = 0.$$

Define $n_1 := ad \in \mathbb{N}\backslash\{0\}$ and $z := \frac{y^{ad}}{q^m} \in \mathbb{C}_p^\times$. Finally

$$x^{n_1} = x^{ad} = \pi(y)^{ad} = \pi\left(y^{ad}\right) = \pi\left(\frac{y^{ad}}{q^m}\right) = \pi(z)$$

and $\operatorname{val}(z) = 0$.

<u>Step 2</u>: Find an $n_2 \in \mathbb{N}\backslash\{0\}$ such that $z^{n_2} = 1 + u$, with $u \in \mathbb{C}_p$.

Consider the reduction map

$$\mathbb{C}_p \xrightarrow{\text{red}} \mathbb{C}_p/\mathbb{C}_p \cong \overline{\mathbb{F}}_p.$$

It holds

$$\mathbb{C}_p \backslash \mathbb{C}_p = \{x \in \mathbb{C}_p |\ |x| = 1\} \xrightarrow{\text{red}} \left(\mathbb{C}_p/\mathbb{C}_p\right)^\times \cong \overline{\mathbb{F}}_p^\times$$

and

$$\overline{\mathbb{F}}_p^{\times} = \bigcup_{k \in \mathbb{N} \setminus \{0\}} \mathbb{F}_{p^k}^{\times}.$$

Let $s \in \overline{\mathbb{F}}_p^{\times}$. Then there exists a $k_s \in \mathbb{N}$ such that $s \in \mathbb{F}_{p^{k_s}}^{\times}$. As $\mathbb{F}_{p^k}^{\times}$ is finite for all $k \in \mathbb{N}$, it is torsion. Hence there exists an $m_s \in \mathbb{N}\setminus\{0\}$ such that $s^{m_s} = 1$.
The z from step 1 has $\mathrm{val}(z) = 0$, and thus $|z| = 1$. This gives $\mathrm{red}(z) \in \overline{\mathbb{F}}_p^{\times}$. It follows that for $\mathrm{red}(z)$ there exists an $n_2 \in \mathbb{N}\setminus\{0\}$ such that

$$\mathrm{red}\left(z^{n_2}\right) = (\mathrm{red}(z))^{n_2} = 1.$$

Lifting this equation gives

$$z^{n_2} = 1 + u$$

with $u \in \mathbb{C}_p$.

Step 3: Find an $n_3 \in \mathbb{N}\setminus\{0\}$ such that $\pi\left((1+u)^{n_3}\right) \in H$.
Claim 1: For $v \in \mathbb{C}_p$ it holds

$$|(1+v)^p - 1| \leq |v| \cdot \max\left(|p|, |v|\right). \tag{4.4}$$

Proof of the claim:

$$(1+v)^p - 1 = \left(\sum_{k=0}^{p} \binom{p}{k} v^k\right) - 1 = \sum_{k=1}^{p} \binom{p}{k} v^k$$

and

$$\begin{aligned}
\left|\sum_{k=1}^{p} \binom{p}{k} v^k\right| &\leq \max_{k=1,\dots,p} \left(\left|\binom{p}{k}\right| \cdot |v|^k\right) \\
&\leq \max\left(\max_{k=1,\dots,p-1} \left(|p| \cdot |v|^k\right), |v|^p\right) \\
&\leq \max\left(|p| \cdot |v|, |v|^2\right) \\
&\leq |v| \cdot \max\left(|p|, |v|\right),
\end{aligned}$$

as $p \geq 2$, $|v| < 1$ and $p \mid \binom{p}{k}$ for $k \neq 0, p$.

Claim 2: For $u \in \mathbb{C}_p$ it holds

$$\left|(1+u)^{p^m} - 1\right| \leq |u| \cdot \max\left(|p|, |u|\right)^m. \tag{4.5}$$

Proof of the claim: By induction.

The case $m=0$ is trivial. Assume that (4.5) holds for $m \in \mathbb{N}$. Define

$$v := (1+u)^{p^m} - 1.$$

From (4.5) it follows $|v| < 1$. Then (4.5) becomes

$$|v| \leq |u| \cdot \max\left(|p|, |u|\right)^m . \tag{4.6}$$

As $|p|, |u| < 1$, it holds $\max\left(|p|, |u|\right) < 1$ and hence

$$|v| \leq |u|. \tag{4.7}$$

We get

$$\begin{aligned}
\left|(1+u)^{p^{m+1}} - 1\right| &= \left|\left((1+u)^{p^m}\right)^p - 1\right| \\
&= |(v+1)^p - 1| \\
&\overset{(4.4)}{\leq} |v| \cdot \max\left(|p|, |v|\right) \\
&\overset{(4.6)}{\leq} |u| \cdot \max\left(|p|, |u|\right)^m \cdot \max\left(|p|, |v|\right) \\
&\overset{(4.7)}{\leq} |u| \cdot \max\left(|p|, |u|\right)^{m+1} .
\end{aligned}$$

This proves Claim 2. With this one gets

$$\left|(1+u)^{p^m} - 1\right| \leq |u| \cdot \max\left(|p|, |u|\right)^m \xrightarrow[m \to +\infty]{} 0$$

and hence

$$(1+u)^{p^m} \xrightarrow[m \to +\infty]{} 1.$$

The lifting $\pi^{-1}(H) \subseteq \mathbb{C}_p^\times$ of H is an open subgroup of $\mathbb{C}_p^\times$, containing the neutral element 1. Hence there exists an $m_0 \in \mathbb{N}\backslash\{0\}$ such that

$$(1+u)^{p^{m_0}} \in \pi^{-1}(H).$$

Defining $n_3 := p^{m_0}$ delivers eventually

$$\pi\left((1+u)^{n_3}\right) \in H.$$

Conclusion:

$$H \ni \pi\left((1+u)^{n_3}\right) = \pi\left((z^{n_2})^{n_3}\right) = \left(\pi\left(z\right)^{n_2}\right)^{n_3} = \left((x^{n_1})^{n_2}\right)^{n_3} = x^{n_1 n_2 n_3} = x^n$$

with $n := n_1 n_2 n_3$. Therefore $A(\mathbb{C}_p)/H$ is a torsion group.

Lemma 4.1.4. *Let G be a K-Lie group. There exists a fundamental system of open neighbourhoods of* 0 *in G, consisting of K-Lie subgroups of G.*

Proof. Proposition 1, Chapter III, §7.1, [Bou98]. ☐

Remark 4.1.5. Notice that Lemma 4.1.4 holds only for the non-archimedean case.

Proof. Consider a real connected Lie group G, for instance $(\mathbb{R}, +)$. Let H be an open Lie subgroup. If $h \in H$, it holds $hH \subseteq H$ because of the group law. For $g \notin H$ one has $gH \cap H = \varnothing$, and hence

$$G \backslash H = \bigcup_{g \notin H} gH,$$

which is an open set, as H, and thus gH, is open. This gives H is closed.
Eventually every open Lie subgroup has to be closed. In the p-adic case this is correct for all the balls around 0 but in the real case this is only true for the whole Lie group G. ☐

Example 4.1.6. (i) $\{B_\epsilon^+(0) | \epsilon > 0\}$ is a fundamental system of open neighbourhoods of 0 in $(\mathbb{Q}_p, +)$, and consists of $\mathbb{Q}_p$-Lie subgroups.

(ii) $\{1 + B_\epsilon^+(0) | \epsilon > 0\}$ is a fundamental system of open neighbourhoods of 1 in $(\mathbb{Q}_p^\times, \cdot)$, and consists of $\mathbb{Q}_p^\times$-Lie subgroups.

(iii) $\{1 + aM_n(\mathbb{Z}_p) | a \in p\mathbb{Z}_p\}$ is a fundamental system of open neighbourhoods of 1 in $(\mathrm{GL}_n(\mathbb{Q}_p), \cdot)$, and consists of $\mathbb{Q}_p^\times$-Lie subgroups.

Lemma 4.1.7. *Let K be a complete subfield of $\mathbb{C}_p$. Furthermore A is an abelian variety of K and $x \in A(K)$. Then there exists a sequence of positive integers $(n_i)_{i \in \mathbb{N}}$ such that*

$$n_i x \xrightarrow[i \to +\infty]{} 0.$$

Proof. One can assume $K = \mathbb{C}_p$. Lemma 4.1.4 allows to work with open K-Lie subgroups instead of open neighbourhoods of 0. The manifold structure of the K-Lie group $A(K)$ provides that $A(K)$ is locally homeomorphic to K^n for some $n \in \mathbb{N}$. Since K^n is a metric space, $A(K)$ accomplishes the first axiom of countability, for instance in 0. Because of this, it is allowed to consider just countably many open neighbourhoods of 0, meaning just countably many open K-Lie subgroups.
Let H_i be an open K-Lie subgroup of $A(K)$. Then

$$[x] \in A(K)/H_i,$$

which is a torsion group because of Theorem 4.1.2. In this way it exists an $n_i \in \mathbb{N} \backslash \{0\}$ such that

$$[n_i x] = n_i [x] = 0.$$

That means $n_i x \in H_i$, and finally

$$n_i x \xrightarrow[i \to +\infty]{} 0.$$

☐

Definition 4.1.8. Let G be a K-Lie group, U an open neighbourhood of 0 in $\mathrm{Lie}(G)$, and

$$\phi : U \longrightarrow G$$

an analytic mapping, such that $\phi(0) = 0$ and $T_0(\phi) = \mathrm{id}_{\mathrm{Lie}(G)}$. Furthermore for all $b \in \mathrm{Lie}(G)$ it holds

$$\phi((\lambda + \lambda')b) = \phi(\lambda b) \cdot \phi(\lambda' b)$$

for all $|\lambda|$ and $|\lambda'|$ sufficiently small.
Then ϕ is called an **exponential mapping of** G.

Lemma 4.1.9. *Let G be a K-Lie group. There exists an exponential mapping ϕ of G with the following properties:*

(i) ϕ is defined on an open subgroup U of the additive group $\mathrm{Lie}(G)$*;*
(ii) $\phi(U)$ is an open subgroup of G and ϕ is an isomorphism of the analytic manifold U onto the analytic manifold $\phi(U)$;
(iii) $\phi(nx) = \phi(x)^n$ for all $x \in U$ and all $n \in \mathbb{Z}$.

Proof. Proposition 3, Chapter III, §7.2, [Bou98]. □

Remark 4.1.10. In contrast to the real (complex) exponential mapping, the non-archimedean counterpart is in general not defined everywhere. Furthermore for a K-Lie group there can exist two different exponential mappings, whereas in the real (complex) case exp is unique.

Example 4.1.11. Consider the $\mathbb{C}_p$-Lie group $\mathbb{C}_p^\times$ with its Lie algebra $\mathbb{C}_p$. Then

$$\begin{aligned} \exp : \mathbb{C}_p &\longrightarrow \mathbb{C}_p \\ x &\longmapsto \sum_{n=0}^{+\infty} \frac{x^n}{n!} \end{aligned}$$

is an exponential mapping of $\mathbb{C}_p$. But it does not converge everywhere. For example in 1 we have

$$\exp(1) = \sum_{n=0}^{+\infty} \frac{1}{n!}$$

and $\left|\frac{1}{n!}\right| \geq 1$, meaning that the series does not converge and $\exp(1)$ is not well-defined. It just converges on the open ball around 0, with radius $p^{-\frac{1}{p-1}}$.

Definition 4.1.12. Let G be a group. Define

$$G_f := \left\{ x \in G \,\middle|\, \exists (n_i)_{i \in \mathbb{N}}, n_i \in \mathbb{N} \backslash \{0\} \text{ s.t. } n_i x \xrightarrow[i \to +\infty]{} 0 \right\}.$$

Example 4.1.13. For $G = \mathbb{C}_p^\times$ it holds $G_f = \{x \in \mathbb{C}_p | \; |x| = 1\}$.

Proof. "$\subseteq$": Let $x \in G_f$, meaning that there exists a series $(n_i)_{i\in\mathbb{N}}$ with $n_i \in \mathbb{N}\backslash\{0\}$ such that

$$x^{n_i} \xrightarrow[i\to+\infty]{} 1$$

which is equivalent to

$$|x^{n_i} - 1| \xrightarrow[i\to+\infty]{} 0. \tag{4.8}$$

Suppose $|x| \neq 1$. Then $|x^{n_i}| = |x|^{n_i} \neq 1$ for all $i \in \mathbb{N}$. Hence

$$|x^{n_i} - 1| = \max(|x^{n_i}|, |1|) \geq 1,$$

which contradicts (4.8).

"$\supseteq$": Let $x \in \mathbb{C}_p$ with $|x| = 1$. Then $x \in \mathbb{C}_p$ and one may reduce x to $\overline{x} \neq \overline{0}$. Assume firstly also $\overline{x} \neq \overline{1}$. Then, by Lemma 5.1.4, there exists exactly one root of unity $r \in \overline{x}$ in this residue class. Defining $m := x - r$ yields $|m| < 1$, as r and x are in the same residue class. Hence x can be written as

$$x = r + m = r(1 + \tfrac{m}{r})$$

with $\left|\frac{m}{r}\right| = \frac{|m|}{|r|} = |m| < 1$.
Since r is a root of unity, there exists a $k \in \mathbb{N}\backslash\{0\}$ such that $r^k = 1$. Furthermore

$$\left|\left(1 + \tfrac{m}{r}\right)^{p^i} - 1\right| \leq |\tfrac{m}{r}| \cdot \max\left(|p|, |\tfrac{m}{r}|\right)^i \xrightarrow[i\to+\infty]{} 0$$

was proven in Step 3 of Example 4.1.3. It is easily shown that Claim 1 of Step 3 in Example 4.1.3 is also true for the exponent kp instead of p. In the proof of claim 2 of Step 3 in 4.1.3 it is just used that $|p| < 1$ and not anymore that p is a prime (this is used in Claim 1). Since $|kp| < 1$, one may deduce

$$\left|\left(1 + \tfrac{m}{r}\right)^{(kp)^i} - 1\right| \leq |\tfrac{m}{r}| \cdot \max\left(|kp|, |\tfrac{m}{r}|\right)^i \xrightarrow[i\to+\infty]{} 0.$$

For $i \neq 0$ it holds

$$\begin{aligned}
\left(1 + \tfrac{m}{r}\right)^{(kp)^i} &= 1 \cdot \left(1 + \tfrac{m}{r}\right)^{(kp)^i} \\
&= \left(r^k\right)^{k^{i-1}p^i} \cdot \left(1 + \tfrac{m}{r}\right)^{(kp)^i} \\
&= r^{(kp)^i} \cdot \left(1 + \tfrac{m}{r}\right)^{(kp)^i} \\
&= \left(r \cdot \left(1 + \tfrac{m}{r}\right)\right)^{(kp)^i} \\
&= x^{(kp)^i}
\end{aligned}$$

Defining $n_i := (kp)^i$ generates a sequence in $\mathbb{N}\backslash\{0\}$ such that

$$|x^{n_i} - 1| \xrightarrow[i\to+\infty]{} 0.$$

In the case of $\overline{x} = \overline{1}$, one takes $r = 1$ and $k = 1$. This shows $x \in G_f$. □

Lemma 4.1.14. *Let the characteristic p of the residue field of K be different from 0, and G be a finite-dimensional K-Lie group. Then G_f is open in G.*

Proof. Proposition 10 (i), Chapter III, §7.6, [Bou98]. □

Lemma 4.1.15. *Let $p \neq 0$, and denote by G a finite dimensional K-Lie group. Then there exists one and only one mapping*

$$\psi : G_f \longrightarrow \mathrm{Lie}(G)$$

with

(i) $\psi(x^n) = n\psi(x)$, for all $x \in G_f$ and $n \in \mathbb{Z}$, and

(ii) it exists an open neighbourhood $V \subseteq G$ of 0, such that $\psi\Big|_V$ is the inverse mapping of an injective exponential mapping.

Proof. Existence of ψ:

For a sufficiently small open subgroup U of $\mathrm{Lie}(G)$, by Lemma 4.1.9(i), there exists an exponential mapping ϕ of G which is defined on U. Assume U so small that $\phi(U) \subseteq G_f$. Then there are open subgroups

$$e \in \phi(U) \subseteq G_f \subseteq G$$

where e is the identity element.

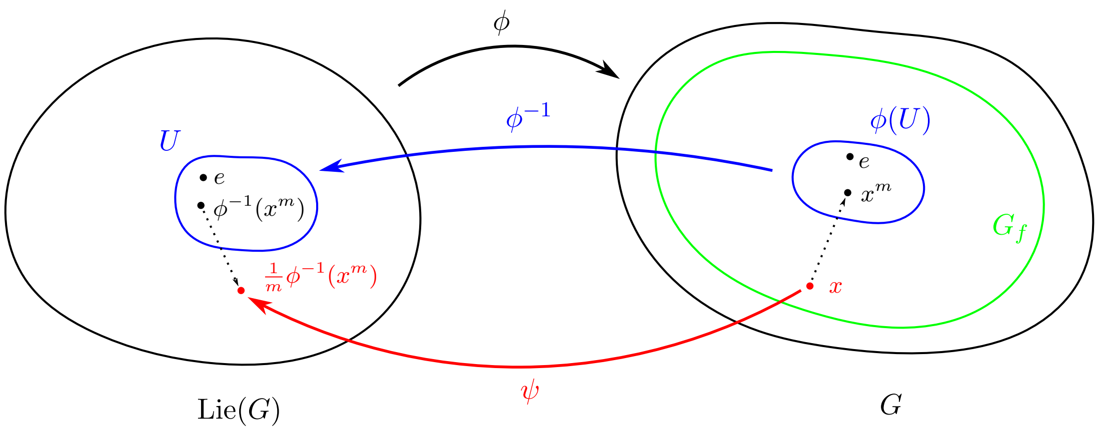


Figure 4.2: Setting

Let $x \in G_f$. By definition of G_f, there exists $m \in \mathbb{N}\backslash\{0\}$ such that $x^m \in \phi(U)$. As ϕ is an isomorphism from U to $\phi(U)$, one gets a unique

$$\phi^{-1}(x^m) \in U \subseteq \mathrm{Lie}(G).$$

The element

$$\frac{1}{m}\phi^{-1}(x^m) \in \mathrm{Lie}(G)$$

is independent of the choice of $m \in \mathbb{N}\backslash\{0\}$. To prove this, assume another $m' \in \mathbb{N}\backslash\{0\}$ such that $x^{m'} \in \phi(U)$. Then $x^{mm'} \in \phi(U)$ and

$$m'\phi^{-1}(x^m) = \phi^{-1}(x^{mm'}) = m\phi^{-1}(x^{m'})$$

which is equivalent to

$$\frac{1}{m}\phi^{-1}(x^m) = \frac{1}{m'}\phi^{-1}(x^{m'}).$$

Define

$$\psi(x) := \frac{1}{m}\phi^{-1}(x^m).$$

<u>Property (i)</u>:

$$\psi(x^n) = \frac{1}{m}\phi^{-1}((x^n)^m) = \frac{1}{m}\phi^{-1}(x^{nm}) = \frac{n}{m}\phi^{-1}(x^m) = n\psi(x)$$

<u>Property (ii)</u>: Clear with $V = \phi(U)$.

<u>Uniqueness of ψ</u>:

Let ψ' be another mapping, fulfilling all the assumptions of Lemma 4.1.15. Let V and V' be the open neighbourhoods, fulfilling property (ii) for ψ respectively ψ'. Denote the inverse exponential mappings by ϕ respectively ϕ' which are defined on $\psi(V)$ respectively $\psi'(V')$. ϕ and ϕ' are both defined on $\psi(V) \cap \psi(V') = \psi(V \cap V')$. Then there exists an open neighbourhood of e such that ϕ and ϕ' coincide. Hence ψ and ψ' coincide on an open neighbourhood W of e, too.
Let $x \in G_f$. There exists $n \in \mathbb{N}\backslash\{0\}$ such that $x^n \in W$. It follows

$$n\psi(x) = \psi(x^n) = \psi'(x^n) = n\psi'(x)$$

which means $\psi = \psi'$. $\square$

Example 4.1.16. In the following it will be tried to investigate ψ with respect to the exponential mapping from Example 4.1.11.
Firstly, from Example 4.1.13 one knows

$$G_f = \{x \in \mathbb{C}_p |\ |x| = 1\},$$

hence

$$\psi : \{x \in \mathbb{C}_p | \ |x| = 1\} \longrightarrow \mathbb{C}_p.$$

The Mercator series

$$\log(1+x) = \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n} x^n,$$

that is defined on the residue class $\overline{1}$, is the inverse of exp from Example 4.1.11. Hence, by property (ii) in Lemma 4.1.15, the map ψ must look like the Mercator series in an open neighbourhood of 1. Requiring property (i) of Lemma 4.1.15, which is a special case of the logarithm law, it is possible to construct uniquely a logarithm on $\{x \in \mathbb{C}_p | \ |x| = 1\}$. This process is described in Construction 5.1.5 for a general logarithm law. But, by Lemma 4.1.15, this leads to the same unique logarithm

$$\log : \{x \in \mathbb{C}_p | \ |x| = 1\} \longrightarrow \mathbb{C}_p.$$

Remark 4.1.17. Note that uniqueness would not be given anymore if we tried to construct log on the whole Lie group $G = \mathbb{C}_p^\times$. This lack of uniqueness corresponds to the branch of the logarithm that is described in Construction 5.1.6. Uniqueness, as e.g. for the abelian logarithm in Theorem 4.1.1, is just given if $G = G_f$.

Corollary 4.1.18. *The map ψ from Lemma 4.1.15 is analytic.*

Proof. In a neighbourhood of $x \in G_f$, the map ψ is composed of the analytic mappings $x \mapsto x^m$, $y \mapsto \phi^{-1}(y)$ and $z \mapsto \dfrac{1}{m} z$, which are analytic. □

Definition 4.1.19. The mapping ψ of Lemma 4.1.15 is called the **logarithmic mapping of** G and denoted by $\log_G$ or simply log.

Lemma 4.1.20. *Assume $p \neq 0$. Let x, y be two permutable elements of G_f. Then $xy \in G_f$ and*

$$\log_G(xy) = \log_G(x) + \log_G(y).$$

Proof. If x^n and y^n tend to 0 for $n \longrightarrow +\infty$, also $(xy)^n = x^n y^n$ tends to 0 for $n \longrightarrow +\infty$. Choose U to be an open subgroup of the additive group $\mathrm{Lie}(G)$ such that $\log_G$, restricted to $\phi(U)$, is the inverse mapping of an exponential mapping ϕ on U. There exists an $n \in \mathbb{N}\backslash\{0\}$ such that $x^n, y^n \in \phi(U)$ because $x, y \in G_f$. Define

$$u := \log_G(x^n) \qquad \text{and} \qquad v := \log_G(y^n).$$

The property $[u, v] = 0$ follows from formula (2) in Proposition 4, Chapter III, §7.2, [Bou98]. Hence one can use the Baker-Campbell-Hausdorff formula

$$\exp(x+y) = \exp(x)\exp(y)\exp\left(-\tfrac{[x,y]}{2}\right)$$

to get

$$\phi(\lambda(u+v)) = \phi(\lambda u)\phi(\lambda v)$$

for $|\lambda|$ sufficiently small. As

$$\left|p^i\right| \xrightarrow[i\to+\infty]{} 0,$$

it holds, for i sufficiently large,

$$\begin{aligned} \phi(p^i(u+v)) &= \phi(p^i u)\phi(p^i v) \\ \phi(p^i(u+v)) &= \phi(u)^{p^i}\phi(v)^{p^i} \\ p^i(u+v) &= \log_G\left(\phi(u)^{p^i}\phi(v)^{p^i}\right) \\ p^i\left(\log_G(x^n)+\log_G(y^n)\right) &= \log_G\left(x^{np^i}y^{np^i}\right) \\ np^i\left(\log_G(x)+\log_G(y)\right) &= np^i\log_G(xy) \end{aligned}$$

and therefore

$$\log_G(xy) = \log_G(x) + \log_G(y)$$

as $np^i \neq 0$. □

The proof of Theorem 4.1.1 is remaining:

Proof. As $A(K)$ is a finite-dimensional K-Lie group and $p \neq 0$, it is possible to apply Lemma 4.1.15 and get a unique mapping

$$\log_A : A(K)_f \longrightarrow \mathrm{Lie}(A).$$

From Lemma 4.1.7 one gets $A(K)_f = A(K)$. With Corollary 4.1.18 and Lemma 4.1.20 it finally follows that

$$\log_A : A(K) \longrightarrow \mathrm{Lie}(A)$$

is a unique homomorphism of K-Lie groups. The property $\mathrm{d}\exp_A = \mathrm{id}$ yields the same for the logarithm, namely

$$\mathrm{d}\log_A = \mathrm{id}\,.$$

□

Remark 4.1.21. Note that the uniqueness of the logarithm on $A(K)$ is just given because of Theorem 4.1.2. Otherwise there would just be a unique logarithm on $A(K)_f$, that might be quite a small subset, as we have seen in Example 4.1.13. This underlines the importance of Theorem 4.1.2, which is an absolutely non-trivial result.

4.2 p-adic abelian integrals on abelian varieties

Lemma 4.2.1. *On an abelian variety any differential one-form is invariant, meaning*

$$\Omega^1_{A/K} = \Omega^1_{\mathrm{inv}}(A).$$

Proof. Proposition 1.5, [GM]. □

Lemma 4.2.2. *One may identify*

$$\Omega^1_{A/K}(A) = \mathrm{Hom}_K(\mathrm{Lie}(A), K).$$

Proof. As a consequence of Lemma 2.5.5, $\mathrm{Lie}(A) = T_0(A)$ is dual to the fibre of 0 over $\Omega_{A/K}$. As all $\omega \in \Omega_{A/K}$ are invariant (Lemma 4.2.1), they are completely determined by this fibre, and it gives the result that $\mathrm{Lie}(A)$ is dual to $\Omega_{A/K}$. □

Definition 4.2.3. For $P \in A(K)$ and $\omega \in \Omega^1_{A/K}$, it will be defined

$$^{\mathrm{Ab}}\!\!\int_0^P \omega = \langle \log_A(P), \omega \rangle$$

where $\langle\ ,\ \rangle$ is the dual pairing between $\mathrm{Lie}(A)$ and $\Omega^1_{A/K}$. Furthermore

$$^{\mathrm{Ab}}\!\!\int_P^Q \omega := {}^{\mathrm{Ab}}\!\!\int_0^Q \omega - {}^{\mathrm{Ab}}\!\!\int_0^P \omega.$$

$^{\mathrm{Ab}}\!\int$ is called the **abelian integral** on A.

Lemma 4.2.4. *Let $P, Q, R \in A(K)$. Then*

$$^{\mathrm{Ab}}\!\!\int_P^R \omega = {}^{\mathrm{Ab}}\!\!\int_P^Q \omega + {}^{\mathrm{Ab}}\!\!\int_Q^R \omega.$$

Proof.

$$\begin{aligned} ^{\mathrm{Ab}}\!\!\int_P^R \omega &= {}^{\mathrm{Ab}}\!\!\int_0^R \omega - {}^{\mathrm{Ab}}\!\!\int_0^P \omega \\ &= {}^{\mathrm{Ab}}\!\!\int_0^Q \omega - {}^{\mathrm{Ab}}\!\!\int_0^P \omega + {}^{\mathrm{Ab}}\!\!\int_0^R \omega - {}^{\mathrm{Ab}}\!\!\int_0^Q \omega \\ &= {}^{\mathrm{Ab}}\!\!\int_P^Q \omega + {}^{\mathrm{Ab}}\!\!\int_Q^R \omega \end{aligned}$$

□

Lemma 4.2.5. *Let A, B be abelian varieties and $f : A(K) \longrightarrow B(K)$ be a homomorphism of K-Lie groups. Then*

$$^{\mathrm{Ab}}\!\!\int_P^Q f^*\omega = {}^{\mathrm{Ab}}\!\!\int_{f(P)}^{f(Q)} \omega$$

for all $P, Q \in A(K)$ and $\omega \in \Omega^1_{B/K}$.

Proof. Prove first

$$(T_0(f))^* \log_B = \log_A,$$

where $T_0(f)$ is the tangent map from Definition 2.2.17, which is often denoted by $\mathrm{d}f$ in the literature.
Every K-Lie group fulfils the definition of a Lie group germ (Definition 5, Chapter III, §1.10, [Bou98]) with its multiplication. This allows to apply Proposition 8, Chapter III, §4.4, [Bou98], which states that the diagram

$$\begin{array}{ccc} A(K) & \xrightarrow{f} & B(K) \\ {\scriptstyle \exp_A}\uparrow & & \uparrow{\scriptstyle \exp_B} \\ \mathrm{Lie}(A) & \xrightarrow{T_0(f)} & \mathrm{Lie}(B) \end{array}$$

commutes on a small open neighbourhood $U \subseteq \mathrm{Lie}(A)$ of 0.
There exist open neighbourhoods $U_A \subseteq A(K)$ and $U_B \subseteq B(K)$ of 0, sufficiently small, such that

$$\log_A \big|_{U_A} : U_A \longrightarrow \mathrm{Lie}(A)$$

respectively

$$\log_B \big|_{U_B} : U_B \longrightarrow \mathrm{Lie}(B),$$

from Theorem 4.1.1, is the inverse mapping of $\exp_A$ respectively $\exp_B$.
Since f and $\log_A$ are homomorphisms of K-Lie groups, it holds $f(0) = 0$ and it is possible to find an open neighbourhood $U' \subseteq U_A$ of 0, such that $f(U') \subseteq U_B$ and $\log_A(U') \subseteq U$. The commutativity of the diagram in the beginning, delivers

$$f \circ \exp_A = \exp_B \circ T_0(f)$$

on U. This can be expanded to

$$\begin{aligned} \log_B \circ f \circ \exp_A \circ \log_A &= \log_B \circ \exp_B \circ T_0(f) \circ \log_A \\ \log_B \circ f &= T_0(f) \circ \log_A, \end{aligned}$$

which holds on U'. The resulting equation may be written in the diagram

$$\begin{array}{ccc} U' \subseteq U_A & \xrightarrow{f} & U_B \\ {\scriptstyle \log_A}\downarrow & & \downarrow{\scriptstyle \log_B} \\ \mathrm{Lie}(A) & \xrightarrow{T_0(f)} & \mathrm{Lie}(B), \end{array}$$

which commutes on the open neighbourhood $U' \subseteq U_A \subseteq A(K)$ of 0. In Lemma 4.1.15, the map $\log_A \big|_{U'}$ was extended to $A(K)$ in the following way: Take an element $x \in A(K)$.

Then there exists an $n \in \mathbb{N}\backslash\{0\}$ such that $x^n \in U'$. Finally, as it was already shown in Lemma 4.1.15, the element x will be mapped to

$$\frac{1}{n} \exp_A^{-1}(x^n) \in \mathrm{Lie}(A)$$

by $\log_A$. Consider the behaviour of $x^n \in U'$ under the following maps:

$$\begin{tikzcd} x^n \arrow[r, mapsto, "f"] \arrow[d, mapsto, "\log_A"'] & f(x^n) \arrow[d, mapsto, "\log_B"] \\ \exp_A^{-1}(x^n) \arrow[r, mapsto, "T_0(f)"] & \exp_B^{-1}(f(x^n)) \end{tikzcd}$$

From the homomorphism property of f and Lemma 4.1.20 follows

$$\exp_A^{-1}(x^n) = \left(\exp_A^{-1}(x)\right)^n$$

and

$$\exp_B^{-1}(f(x^n)) = \exp_B^{-1}((f(x))^n) = \left(\exp_B^{-1}(f(x))\right)^n .$$

Rewriting the commutative diagram gives

$$\begin{tikzcd} x^n \arrow[r, mapsto, "f"] \arrow[d, mapsto, "\log_A"'] & (f(x))^n \arrow[d, mapsto, "\log_B"] \\ \left(\exp_A^{-1}(x)\right)^n \arrow[r, mapsto, "T_0(f)"] & \left(\exp_B^{-1}(f(x))\right)^n . \end{tikzcd}$$

After cancelling n and writing log instead of $\exp^{-1}$, when applied to x, it remains

$$\begin{tikzcd} x \arrow[r, mapsto, "f"] \arrow[d, mapsto, "\log_A"'] & f(x) \arrow[d, mapsto, "\log_B"] \\ \log_A(x) \arrow[r, mapsto, "T_0(f)"] & \log_B(f(x)), \end{tikzcd}$$

meaning that the diagram commutes for any $x \in A(K)$, or, in other words,

$$\begin{tikzcd} A(K) \arrow[r, "f"] \arrow[d, "\log_A"'] & B(K) \arrow[d, "\log_B"] \\ \mathrm{Lie}(A) \arrow[r, "T_0(f)"] & \mathrm{Lie}(B) \end{tikzcd}$$

commutes on the whole K-Lie group $A(K)$.
This yields the property

$$(T_0(f))^* \log_B = \log_A$$

and finally

$$\begin{aligned}
{}^{\mathrm{Ab}}\!\!\int_0^P f^*\omega &= \langle \log_A(P), f^*\omega\rangle \\
&= \langle ((T_0(f))^* \log_B)(P), f^*\omega\rangle \\
&= \langle \log_B(f(P)), \omega\rangle \\
&= {}^{\mathrm{Ab}}\!\!\int_0^{f(P)} \omega.
\end{aligned}$$

□

Proposition 4.2.6. *The map*

$$\begin{aligned}
\sigma : A(K) &\longrightarrow K \\
P &\longmapsto {}^{\mathrm{Ab}}\!\!\int_0^P \omega
\end{aligned}$$

is a morphism of K-Lie groups.

Proof. That σ is a K-analytic function is clear.

$$\begin{aligned}
\sigma(P+Q) &= {}^{\mathrm{Ab}}\!\!\int_0^{P+Q} \omega \\
&= \langle \log_A(P+Q), \omega\rangle \\
&= \langle \log_A(P) + \log_A(Q), \omega\rangle \\
&= \langle \log_A(P), \omega\rangle + \langle \log_A(Q), \omega\rangle \\
&= \sigma(P) + \sigma(Q)
\end{aligned}$$

as $\log_A$ is a homomorphism of K-Lie groups. Therefore σ is a morphism of K-Lie groups. □

4.3 p-adic abelian integrals on curves

In this section a smooth, proper, connected $\mathbb{C}_p$-curve X will be fixed. Let J be the Jacobian of X, that is an abelian variety over $\mathbb{C}_p$.

Definition 4.3.1. Fix a base point $P_0 \in X(\mathbb{C}_p)$, and let $\iota : X \longrightarrow J$ be the Abel-Jacobi map with respect to P_0. Then its pullback ι^* is an isomorphism between $\Omega^1_{J/\mathbb{C}_p}$ and $\Omega^1_{X/\mathbb{C}_p}$. For $P, Q \in X(\mathbb{C}_p)$ and $\omega \in \Omega^1_{X/\mathbb{C}_p}$, the **abelian integral** is defined by

$$ {}^{\mathrm{Ab}}\!\!\int_P^Q \omega := {}^{\mathrm{Ab}}\!\!\int_{\iota(P)}^{\iota(Q)} (\iota^*)^{-1}\omega. $$

Lemma 4.3.2. *The abelian integral is independent of the choice of the base point $P_0 \in X(\mathbb{C}_p)$.*

Proof. A change of the base point P_0 would create just a translation of ι^*. By Lemma 4.2.1, all the differential one-forms $\Omega^1_{J/\mathbb{C}_p}$ are invariant. Hence

$$(\iota^*)^{-1}\omega \in \Omega^1_{J/\mathbb{C}_p}$$

in Definition 4.3.1 is well-defined, which yields the claim. □

Theorem 4.3.3. *The abelian integral on curves satisfies the following properties:*

(i) It is path-independent.
(ii) For $P_1, P_2, P_3 \in X(\mathbb{C}_p)$ and $\omega \in \Omega^1_{X/\mathbb{C}_p}(X)$, it holds

$$^{\mathrm{Ab}}\!\int_{P_1}^{P_3} \omega = {}^{\mathrm{Ab}}\!\int_{P_1}^{P_2} \omega + {}^{\mathrm{Ab}}\!\int_{P_2}^{P_3} \omega.$$

(iii) For fixed $P, Q \in X(\mathbb{C}_p)$, the map $\omega \longmapsto {}^{\mathrm{Ab}}\!\int_P^Q \omega$ is $\mathbb{C}_p$-linear in ω.
(iv) Within an open ball in the semistable decomposition (defined Definition 2.6.5), ${}^{\mathrm{Ab}}\!\int_P^Q \omega$ is calculated by formal antidifferentiation of ω.

Proof. (i) ${}^{\mathrm{Ab}}\!\int_P^Q \omega$ makes no reference to a path, meaning that it is path-independent.
(ii) Follows from Lemma 4.2.4.
(iii) Let $\lambda \in \mathbb{C}_p$. Then

$$\begin{aligned}
\lambda\omega \longmapsto {}^{\mathrm{Ab}}\!\int_P^Q \lambda\omega &= {}^{\mathrm{Ab}}\!\int_{\iota(P)}^{\iota(Q)} (\iota^*)^{-1}(\lambda\omega) \\
&= {}^{\mathrm{Ab}}\!\int_{\iota(P)}^{\iota(Q)} \lambda\left((\iota^*)^{-1}\omega\right) \\
&= \lambda \cdot {}^{\mathrm{Ab}}\!\int_{\iota(P)}^{\iota(Q)} (\iota^*)^{-1}\omega \\
&= \lambda \cdot {}^{\mathrm{Ab}}\!\int_P^Q \omega
\end{aligned}$$

as ι^* is a $\mathbb{C}_p$-isomorphism. The linearity of the abelian integral for abelian varieties comes from the bilinearity of the dual pairing in Definition 4.2.3.
(iv) Follows from Corollary 6.3.4 and the fact that the Berkovich-Coleman integral (chapter 5) is calculated by formal antidifferentation on open balls.

□

Corollary 4.3.4. *The abelian integral is an integration theory in the sense of Definition 3.2.1.*

Proof. This is a direct consequence of Theorem 4.3.3. □

Remark 4.3.5. Notice that the path-independency is in general not given in the complex case. For example on a torus (Figure 4.3) not every integral from a point P to Q is independent from the path γ between these points.

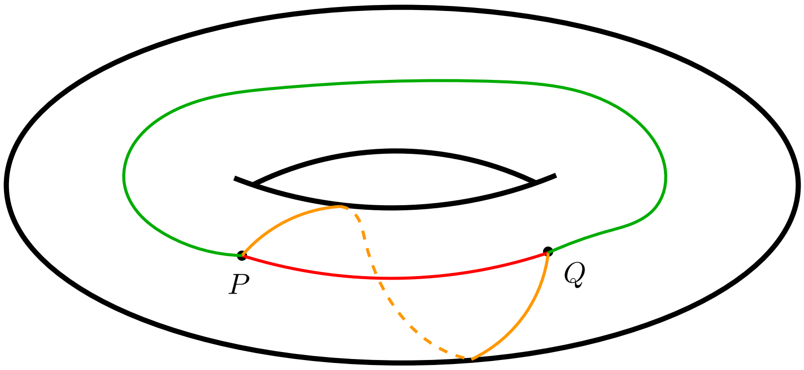


Figure 4.3: From these three paths, three different integrals may arise

5 Berkovich-Coleman integral

Robert F. Coleman constructed in the 1980s, starting with his paper *Dilogarithms, regulators and p-adic L-functions*, [Col82], a theory of p-adic integration on rigid spaces which admit an admissible covering by basic wide open subsets of $\mathbb{P}^1$. This was before Vladimir G. Berkovich developed a more modern language in non-archimedean geometry, named after him. For this reason the results of Coleman will be translated in Berkovich language.

At a point when rigid analysis was almost completely developed, Berkovich formulated the definition of Berkovich spaces and created therewith a totally new field of research in non-archimedean geometry. With this new techniques, he generalized this p-adic integration theory to Berkovich analytic spaces in his book *Integration of One-forms on P-adic analytic spaces*, [Ber07]. This means that, when using the modern theory of Berkovich, one is no longer restricted to the use of curves.

Let K be a field that is algebraically closed and complete with respect to a nontrivial, non-archimedean valuation $\mathrm{val} : K \longrightarrow \mathbb{R} \cup \{\infty\}$.

5.1 Historical approach by Coleman

First the original ideas of Robert F. Coleman are described, using Berkovich theory. Let X be a smooth, proper, connected $\mathbb{C}_p$-curve, along with a semistable model $\mathcal{X}$, as it was already described in the introduction. This gives the following setting.

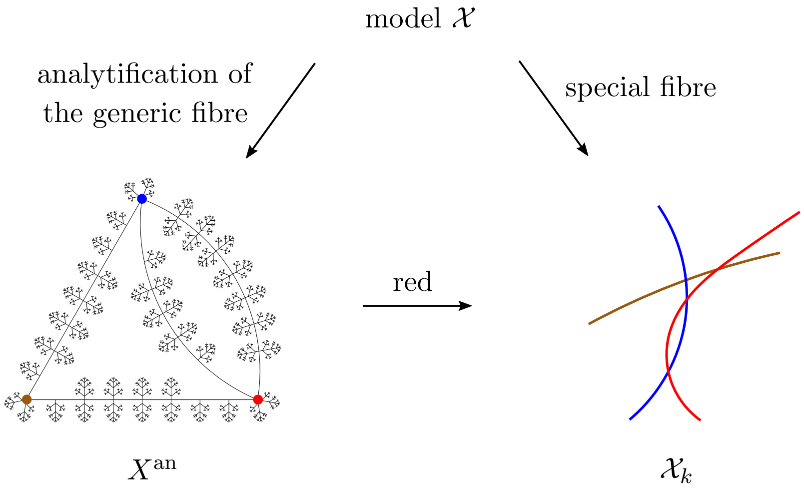


Figure 5.1: Interplay of the model $\mathcal{X}$ and the curve X

By considering the analytification X^{an}, one can omit the totally disconnected topology of the curve X. On top of that, additional information about X^{an} is gained via the model $\mathcal{X}$, which creates an interplay of X^{an} and $\mathcal{X}_k$ via the reduction map. A result of Berkovich and Bosch-Lütkebohmert (Theorem 3.2.4, [KRZ16a]) states that there is the following 1:1 correspondence via the reduction map

X^{an}	$\mathcal{X}_k$
singleton (called vertex)	generic point
open ball isomorphic to $B(1)_+$	smooth point
open annulus isomorphic to $S(\rho)_+$	double point

where

$$B(1)_+ := \{||.|| \in \mathbb{A}^{1,\mathrm{an}} |\ ||T|| < 1\}$$

and

$$S(\rho)_+ := \{||.|| \in \mathbb{A}^{1,\mathrm{an}} |\ \rho < ||T|| < 1\}$$

with $\rho \in (0,1)$. This subdivision of X^{an}, via the reduction map into the sets in the left column, is called semistable decomposition (see Definition 2.6.5).

5.1.1 Branch of the logarithm

On the analytification X^{an} of the curve X it is possible to consider paths, as it is path-connected, in contrast to X. The Berkovich-Coleman integration theory will be restricted to paths with $\mathbb{C}_p$-rational end points. These lie always at the ends of the Berkovich trees. A simple case is the following:

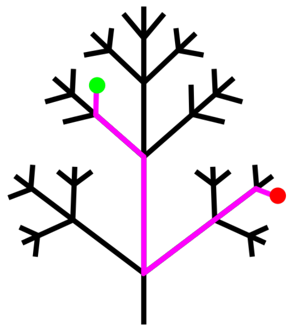


Figure 5.2: Integration between two points within an open ball

Integrating between two points in the same open ball is rather easy, as one can apply the Poincaré lemma. This means there exists an analytic function F with $\omega = \mathrm{d}F$ and it is possible to calculate the integral

$$\int_P^Q \omega = \int_P^Q \mathrm{d}F = F(Q) - F(P).$$

If one wants to integrate between two points of different open balls in the semistable decomposition, it is not possible to cover the path by open balls, because the vertex that has to be passed does not lie in any open ball.

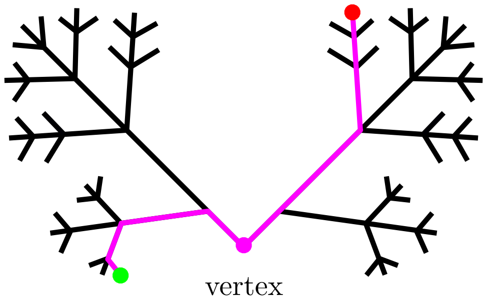


Figure 5.3: Integration between two points in different residue classes

Hence the differential one-form ω is in general not exact on the whole path. There exist analytic functions F_1 and F_2, such that $\omega = \mathrm{d}F_1$ and $\omega = \mathrm{d}F_2$ on the single open balls, which make it possible to calculate $F_1(P)$ and $F_2(Q)$, but the difference $F_2(Q) - F_1(P)$ is not well-defined, as the integral functions F_1 and F_2 are just defined up to a constant. Let us now turn our attention to the open annuli in the semistable decomposition:

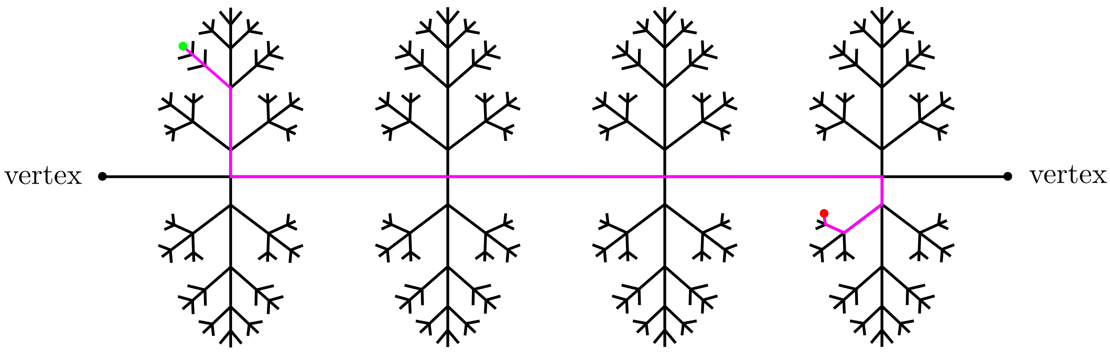


Figure 5.4: Integration between two points within an open annulus

As this is not a disc, it is not possible to apply the Poincaré lemma on the whole annulus. True, on subsets which are open balls one could apply the Poincaré lemma again, but the aim is to define the integration in general on the whole open annulus. The analytic functions on $S(\rho)_+$ are all infinite-tailed Laurent series $f \in \mathbb{C}_p[[T, T^{-1}]]$ that converge for all $T \in \mathbb{C}_p$ with $\rho < |T| < 1$. By Example 2.5.8, any differential one-form ω on $S(\rho)_+$

can be expanded as a Laurent series $f \in \mathbb{C}_p[[T, T^{-1}]]$ that converges for all $T \in \mathbb{C}_p$ with $\rho < |T| < 1$, meaning

$$\omega = f(T)\,\mathrm{d}T = \sum_{n=-\infty}^{+\infty} a_n T^n\,\mathrm{d}T.$$

By Example 2.5.9, one can formally integrate this differential one-form to F

$$\omega = f(T)\,\mathrm{d}T = \mathrm{d}F(T)$$

with

$$F(T) = \sum_{n=-\infty}^{+\infty} \frac{a_n}{n+1} T^{n+1}$$

only if $a_{-1} = 0$.
Hence it is necessary to find a way to integrate $\frac{\mathrm{d}T}{T}$. In the complex setting, the integral of this term is the logarithm, that is not even in $\mathbb{C}$ unique. But for the complex integral it does not matter which branch of the logarithm is chosen, as long as the path is containend in its domain.
To solve this problem, a logarithm for $\mathbb{C}_p$ has to be found.

Definition 5.1.1. The logarithm on $B_1(1) = \left\{x \in \mathbb{C}_p \middle| |x-1| < 1\right\}$ is defined by the Mercator series

$$\log(1+x) = \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n} x^n.$$

Lemma 5.1.2. *The series from Definition 5.1.1 converges on*

$$B_1(1) = \left\{x \in \mathbb{C}_p \middle| |x-1| < 1\right\}.$$

Proof. It holds

$$\left|\frac{(-1)^{n+1}}{n} x^n\right| = \frac{|x^n|}{|n|} = \frac{1}{|n|} \cdot |x|^n$$

with $\frac{1}{|n|} \leq n$ for $n \in \mathbb{N}\backslash\{0\}$ and $|x| < 1$. As the first factor grows at most linearly with n, and the second factor reduces exponentially to 0, the limit is

$$\lim_{n \to +\infty} \frac{1}{|n|} \cdot |x|^n = 0.$$

Thus the series converges on $B_1(1)$. ∎

Lemma 5.1.3. *The formal derivative of* $\log(x)$*, defined by the Mercator series, is* $\frac{1}{x}$*.*

Proof. Formal differentiation for $x \in B_1(1)$ gives

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}x} \log(x) &= \tfrac{\mathrm{d}}{\mathrm{d}x} \sum_{n=1}^{+\infty} \tfrac{(-1)^{n+1}}{n}(x-1)^n \\
&= \sum_{n=1}^{+\infty} \tfrac{\mathrm{d}}{\mathrm{d}x}\left(\tfrac{(-1)^{n+1}}{n}(x-1)^n\right) \\
&= \sum_{n=1}^{+\infty} (-1)^{n+1}(x-1)^{n-1} \\
&= \sum_{n=0}^{+\infty} (1-x)^n \\
&= \tfrac{1}{1-(1-x)} \\
&= \tfrac{1}{x}.
\end{aligned}$$

The formal differentiate converges on $B_1(1)$, as $|x-1| < 1$. In the end the geometric series was used. □

Lemma 5.1.4. *In every residue class* $\neq \overline{0}, \overline{1}$ *of* $\overline{\mathbb{F}}_p$ *there exists exactly one root of unity.*

Proof. Existence: Let $\overline{s} \in \overline{\mathbb{F}}_p$, with $\overline{s} \neq \overline{0}, \overline{1}$ and $s \in \mathbb{C}_p$. $|s| < 1$ would mean $s \in \mathbb{C}_p$ and thus $\overline{s} = \overline{0}$. Therefore it may be assumed $|s| = 1$, meaning that $s \in \left(\mathbb{C}_p\right)^{\times}$. Since $\overline{s}$ is in $\overline{\mathbb{F}}_p$, it is algebraic over $\mathbb{F}_p$, meaning that s is contained in a finite extension F_{ext} of $\mathbb{F}_p$. As $\mathbb{F}_p$ is already finite, the extension must also be finite. The same holds for its multiplicative group, which means that there exists an $n \in \mathbb{N}\backslash\{0\}$ such that

$$\overline{s}^n = \overline{1}.$$

Take the smallest n. This n is unique, as, for any bigger extension, all powers of $\overline{s}$ still remain in F_{ext}. Thus there exists an $m \in \mathbb{C}_p$ such that

$$s^n = 1 + m,$$

meaning that the polynomial equation

$$T^n - (1+m) = 0 \tag{5.1}$$

with the variable T is solvable over $\mathbb{C}_p$, for instance with s. Any solution of (5.1) must have absolute value 1 because $|1+m| = 1$. Hence the solutions $\alpha_1, ..., \alpha_n$ are all contained in $\mathbb{C}_p$, and (5.1) decomposes into

$$(T-\alpha_1)\cdots(T-\alpha_n) = 0 \tag{5.2}$$

as $\mathbb{C}_p$ is algebraically closed. Reducing (5.1) delivers the equation

$$T^n - \overline{1} = \overline{0} \tag{5.3}$$

in $\overline{\mathbb{F}}_p$. A lift to $\mathbb{C}_p$ is

$$T^n - 1 = 0, \tag{5.4}$$

which decomposes into

$$(T - \beta_1) \cdots (T - \beta_n) = 0 \tag{5.5}$$

with $\beta_1, ..., \beta_n \in \mathbb{C}_p$. As the reductions of (5.1) and (5.4) are the same, the reductions of their decompositions (5.2) and (5.5) must also coincide. Hence the $\overline{\alpha}_i$ and $\overline{\beta}_j$ are equal up to permutation for $i, j \in \{1, ..., n\}$.
Since $\overline{s} \in \overline{\mathbb{F}}_p$ solves (5.3), there exists an $i \in \{1, ..., n\}$ such that $\overline{s} = \overline{\alpha}_i$. It has just been shown that for any $i \in \{1, ..., n\}$ there exists a $j \in \{1, ..., n\}$ such that $\overline{\alpha}_i = \overline{\beta}_j$. Therefore

$$\overline{s} = \overline{\beta}_j$$

where $\beta_j \in \mathbb{C}_p$ solves (5.2), meaning $\beta_j^n = 1$. Hence β_j is a root of unity in the residue class of s. This shows existence.

Uniqueness: Consider the residue class $\overline{s} \in \overline{\mathbb{F}}_p$ with $\overline{s} \neq \overline{0}, \overline{1}$ and $s \in \left(\mathbb{C}_p\right)^\times$. Assume that there are two different roots of unity $\zeta_1, \zeta_2 \in \overline{s}$, with $\zeta_1^{n_1} = 1$ and $\zeta_2^{n_2} = 1$ for $n_1, n_2 \in \mathbb{N} \backslash \{0\}$. With $n := n_1 \cdot n_2$ it may be calculated

$$\zeta_1^n = \zeta_1^{n_1 \cdot n_2} = (\zeta_1^{n_1})^{n_2} = 1^{n_2} = 1$$

and

$$\zeta_2^n = \zeta_2^{n_1 \cdot n_2} = (\zeta_2^{n_2})^{n_1} = 1^{n_1} = 1.$$

Therefore both are n-th roots of unity where n does not have to be minimal. Both solve the polynomial equation

$$T^n - 1 = 0, \tag{5.6}$$

meaning that (5.6) decomposes into

$$(T - \zeta_1) \cdot (T - \zeta_2) \cdots (T - \zeta_n) = 0 \tag{5.7}$$

with $\zeta_3, ..., \zeta_n \in \mathbb{C}_p$. The reduction of (5.6) and (5.7) delivers

$$T^n - \overline{1} = \overline{0} \tag{5.8}$$

and

$$(T - \overline{\zeta_1}) \cdot (T - \overline{\zeta_2}) \cdots (T - \overline{\zeta_n}) = 0 \tag{5.9}$$

with $\overline{\zeta_1} = \overline{\zeta_2} = \overline{s}$, meaning that (5.8) and (5.9) have multiple zeroes in $\overline{\mathbb{F}}_p$.
Let $n = p^k \cdot n'$ be the unique decomposition with $p \nmid n'$ and $k \in \mathbb{N}$. Then ζ_1 and ζ_2 can be decomposed into

$$\zeta_1 = \zeta_1' \cdot \zeta_1'' \text{ and } \zeta_2 = \zeta_2' \cdot \zeta_2''$$

such that ζ_1' and ζ_2' are p^k-th and ζ_1'' and ζ_2'' are n'-th roots of unity. It holds $\overline{\zeta_1}' \in \overline{\mathbb{F}}_p$, hence there exists a $k' \in \mathbb{N}\backslash\{0\}$ such that $\overline{\zeta_1}' \in \mathbb{F}_{p^{k'}} \subseteq \mathbb{F}_{p^{k \cdot k'}}$. Since $(\zeta_1')^{p^k} = 1$, also $(\zeta_1')^{p^{k \cdot k'}} = 1$. Considering its reduction, which is contained in $\mathbb{F}_{p^{k \cdot k'}}$, gives

$$\left(\overline{\zeta_1}'\right)^{p^{k \cdot k'}} = \overline{1}.$$

The Frobenius automorphism $x \longmapsto x^{p^{k \cdot k'}}$, which is the identity on $\mathbb{F}_{p^{k \cdot k'}}$, delivers

$$\left(\overline{\zeta_1}'\right)^{p^{k \cdot k'}} = \overline{\zeta_1}'.$$

Hence $\overline{\zeta_1}' = \overline{1}$ which holds for $\overline{\zeta_2}'$ as well. This yields

$$\overline{\zeta_1} = \overline{\zeta_1}' \cdot \overline{\zeta_1}'' = \overline{\zeta_1}'' \text{ and } \overline{\zeta_2} = \overline{\zeta_2}' \cdot \overline{\zeta_2}'' = \overline{\zeta_2}''.$$

Therefore one may consider the equations (5.8) and (5.9) for $n = n'$. Then (5.8) cannot have multiple zeroes as the formal derivative

$$\overline{n'} T = \overline{0}$$

only has $\overline{0}$ as a zero. Its reason is that $\overline{n'} \neq \overline{0}$ for $p \nmid n'$. Therefore $\overline{\zeta_1}'' \neq \overline{\zeta_2}''$ and consequently $\overline{\zeta_1} \neq \overline{\zeta_2}$, which contradicts $\zeta_1, \zeta_2 \in \overline{s}$. This shows uniqueness. Finally in any residue class $\neq \overline{0}, \overline{1}$ of $\overline{\mathbb{F}}_p$ there exists exactly one root of unity. □

Construction 5.1.5. The goal is to extend the definition of the logarithm uniquely to

$$\left(\mathbb{C}_p\right)^{\times} = \left\{x \in \mathbb{C}_p \middle| |x| = 1\right\}.$$

$B_1(1) = 1 + \mathbb{C}_p = 1 + \left\{x \in \mathbb{C}_p \middle| |x| < 1\right\}$ is a subset of $\left(\mathbb{C}_p\right)^{\times}$, as for $m \in \mathbb{C}_p$ holds

$$|1 + m| = \max\left(|1|, |m|\right) = 1.$$

Therefore the term extension is justified.
The aim is to define the logarithm for an arbitrary $s \in \left(\mathbb{C}_p\right)^{\times}$, meaning that it holds already $\overline{s} \neq \overline{0}$. Assume furthermore $\overline{s} \neq \overline{1}$, as the logarithm is already defined for this

residue class. By Lemma 5.1.4 there exists a unique root of unity $r \in \overline{s}$ with the described properties. Defining

$$m := s - r$$

gives $|m| < 1$, as r and s are in the same residue class. By requiring the logarithm law

$$\log(a \cdot b) = \log(a) + \log(b)$$

also for the p-adic logarithm, one can extend log to $\left(\mathbb{C}_p\right)^{\times}$:

$$\begin{aligned} \log(s) &= \log(r + m) \\ &= \log\left(r\left(1 + \tfrac{m}{r}\right)\right) \\ &:= \log(r) + \log\left(1 + \tfrac{m}{r}\right) \end{aligned}$$

The expected logarithm law delivers

$$n \cdot \log(r) = \log(r^n) = \log(1) = 0,$$

which means $\log(r) = 0$. Besides $\log\left(1 + \frac{m}{r}\right)$ is already well-defined by the Mercator series as $\left|\frac{m}{r}\right| = \frac{|m|}{|r|} = |m| < 1$.
After all, a unique p-adic logarithm is defined on $\left(\mathbb{C}_p\right)^{\times}$.

Construction 5.1.6. The next goal is to extend this logarithm to $\mathbb{C}_p^{\times}$. Let $c \in \mathbb{C}_p^{\times}$. It can be uniquely written as

$$c = p^x u$$

with $x = \mathrm{val}(c) \in \mathbb{Q}$ and $u \in \left(\mathbb{C}_p\right)^{\times}$. If, for a logarithm

$$\mathrm{Log} : \mathbb{C}_p^{\times} \longrightarrow \mathbb{C}_p,$$

again the logarithm law should hold, one gets

$$\begin{aligned} \mathrm{Log}(c) &= \mathrm{Log}(p^x u) \\ &= \mathrm{Log}(p^x) + \mathrm{Log}(u) \\ &= x \cdot \mathrm{Log}(p) + \log(u), \end{aligned}$$

as $u \in \left(\mathbb{C}_p\right)^{\times}$ and the logarithm is well-defined on $\left(\mathbb{C}_p\right)^{\times}$. For $\mathrm{Log}(p)$, a value from $\mathbb{C}_p$ has to be chosen.

Lemma 5.1.7. *After choosing a value* $\mathrm{Log}(p) \in \mathbb{C}_p$*, the map*

$$\mathrm{Log} : \mathbb{C}_p^{\times} \longrightarrow \mathbb{C}_p$$

is well-defined and extends log.

Proof. The decomposition $c = p^x u$ is unique, as $x = \mathrm{val}(c)$ is unique. Thus

$$\mathrm{Log}(c) := x \cdot \mathrm{Log}(p) + \log(u)$$

is well-defined.
If $c \in \left(\mathbb{C}_p\right)^{\times}$, we have $x = \mathrm{val}(c) = 0$. Hence $c = p^0 u = u$ and

$$\mathrm{Log}(c) = 0 \cdot \mathrm{Log}(p) + \log(u) = \log(c).$$

□

This leads to the following definition:

Definition 5.1.8. A **branch of the logarithm** is a group homomorphism

$$\mathrm{Log} : \mathbb{C}_p^{\times} \longrightarrow \mathbb{C}_p$$

that restricts to log on $\left(\mathbb{C}_p\right)^{\times}$.

Remark 5.1.9. Similar to the complex case, this p-adic logarithm is not unique. There are infinitely many branches of this logarithm, namely for any choice of $\mathrm{Log}(p) \in \mathbb{C}_p$. As in the complex case it is only important to work all time with the same branch and do not change it within one integral. Therefore one has to choose a certain branch of the logarithm in the beginning and fix it.

Lemma 5.1.10. Log *is analytic on* $B_{|x|}(x)$ *for all* $x \in \mathbb{C}_p^{\times}$.

Proof. Fix $x \in \mathbb{C}_p^{\times}$. Then

$$\mathrm{Log}(x \cdot z) = \mathrm{Log}(x) + \mathrm{Log}(z)$$

is analytic as a function of z on $B_1(1)$. Hence Log is analytic on $xB_1(1) = B_{|x|}(x)$. □

Theorem 5.1.11. $\mathrm{d}\,\mathrm{Log}(z) = \frac{\mathrm{d}z}{z}$ *on* $\mathbb{C}_p^{\times}$ *by formal derivation.*

Proof. Fix $x \in \mathbb{C}_p^{\times}$. The group homomorphism Log will be derivated in the neighbourhood $B_{|x|}(x)$ of x. Consider Log as a function of z.

$$\begin{aligned}
\mathrm{d}\,\mathrm{Log}(z) &= \mathrm{d}\,\mathrm{Log}\left(x \cdot \tfrac{z}{x}\right) \\
&= \mathrm{d}\,\mathrm{Log}(x) + \mathrm{d}\,\mathrm{Log}\left(\tfrac{z}{x}\right) \\
&= 0 + \mathrm{d}\log\left(\tfrac{z}{x}\right) \\
&= \frac{1}{\frac{z}{x}}\,\mathrm{d}\left(\tfrac{z}{x}\right) \\
&= \frac{1}{\frac{z}{x}}\frac{1}{x}\,\mathrm{d}z \\
&= \frac{\mathrm{d}z}{z}.
\end{aligned}$$

As $\frac{z}{x} \in B_1(1)$, it was possible to use Lemma 5.1.3. □

Remark 5.1.12. Berkovich extended this branch of the logarithm to

$$\mathbb{G}_m^{\mathrm{an}} = \operatorname{Spec}(\mathbb{C}_p[T, T^{-1}])^{\mathrm{an}}$$

without its skeleton. See Lemma 1.4.1 (ii), [Ber07]. As any open annulus in the semistable decomposition is a subset of $\mathbb{G}_m^{\mathrm{an}}$, one has an antiderivative of $\frac{\mathrm{d}T}{T}$ on any open annulus without its skeleton.

Remark 5.1.13. With Theorem 5.1.11 it is possible to formally antidifferentiate any Laurent series on $\mathbb{C}_p^{\times}$. Due to Remark 5.1.12 one could define a p-adic integration theory that works on open annuli. This is already more powerful than just integrating on open balls. But still it is not possible to integrate between $\mathbb{C}_p$-points in different open annuli or different open balls that are not contained in an open annulus. For this reason Coleman covered the curve by so called basic wide opens, which he defined using rigid analysis. The following section is describing this, using the more modern language of Berkovich.

5.1.2 Basic wide open subdomains

Take the same assumptions as in the previous section.

Remark 5.1.14. In section 5.1.1 it has already been noted that the main problem lies in passing the vertices in the analytification X^{an}. The first problem focused on will be passing a single vertex. An example of such a path is sketched in the following image:

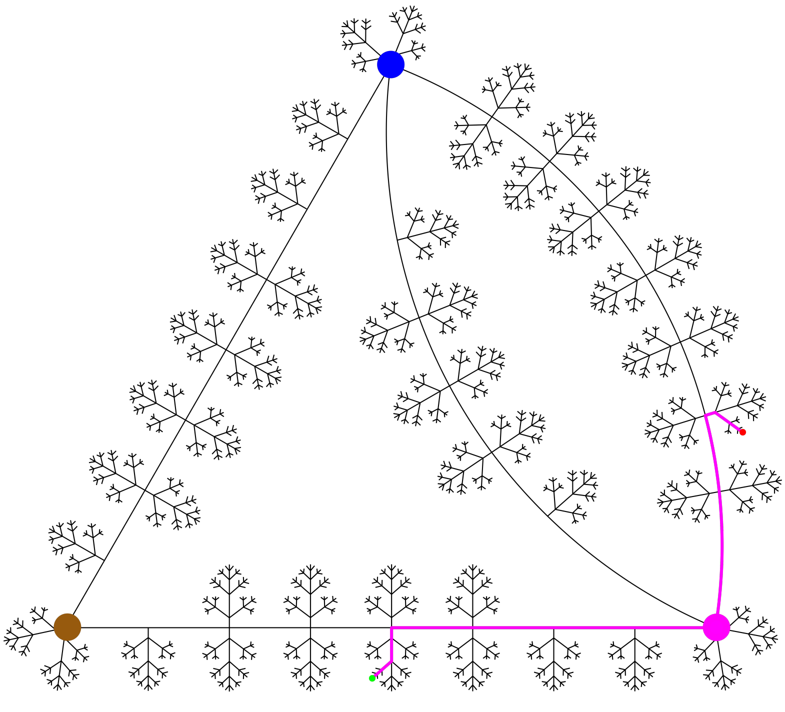


Figure 5.5: A path passing a vertex

Again one could integrate any differential one-form ω on the annulus of the small green point and on the annulus of the small red point. This results in two antiderivatives F_1 and F_2, which are uniquely defined up to a constant. Hence the difference

$$\int_P^Q \omega = F_2(Q) - F_1(P)$$

is again not well-defined. But, in the 1980s, Coleman found a way to integrate ω on both annuli simultaneously such that the above difference becomes well-defined. This only works if both annuli are adjacent to the same vertex. For this reason he defined basic wide open subdomains, which may be considered as an open neighbourhood of a vertex in X^{an}.

Definition 5.1.15. Let $\zeta \in X^{\mathrm{an}}$ be a vertex and I_ζ be the index set of all open edges e_i adjacent to ζ. For each open edge e_i, adjacent to ζ, e_i' is an open interval contained in e_i whose length is contained in $\mathrm{val}(\mathbb{C}_p^\times) = \mathbb{Q}$. Furthermore e_i' has to be adjacent to ζ. Then an **open star neighbourhood** of ζ is defined as the union

$$S_\zeta = \{\zeta\} \cup \bigcup_{i \in I_\zeta} e_i'.$$

Definition 5.1.16. The pre-image of S_ζ under the retraction map τ is called a **basic wide open subdomain** U_ζ of X^{an}.

Remark 5.1.17. In this paper it is sufficient to consider $e_i' = e_i$ in Definition 5.1.15. Hence a basic wide open subdomain becomes the union of the vertex ζ and all the open annuli and open balls adjacent to ζ. Furthermore it is simply-connected because no loops are contained anymore. For the following it will be always presumed $e_i' = e_i$ for basic wide open subdomains if inequality is not explicitely allowed.

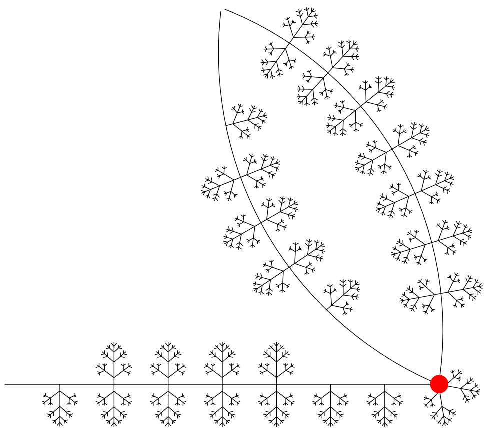


Figure 5.6: Basic wide open subdomain of the red vertex

Definition 5.1.18. For a basic wide open subdomain U_ζ, the vertex ζ is called the **central point** of U_ζ and $\tau^{-1}(\zeta)$ is called the **underlying affinoid** of U_ζ.

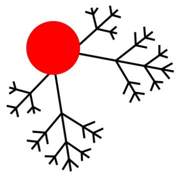


Figure 5.7: Underlying affinoid of the basic wide open subdomain in Figure 5.6

Remark 5.1.19. Consider an irreducible component in $\mathcal{X}_k$ (for example the red one in Figure 5.1). It contains one generic point, whose pre-image under reduction is a vertex ζ, smooth points, whose pre-images are open balls adjacent to ζ, and double points, whose pre-images are open annuli adjacent to ζ. Hence the pre-image of an irreducible component in $\mathcal{X}_k$ under reduction is a basic wide open subdomain of X^{an}, in this example, the one from Figure 5.6.

Remark 5.1.20. Denote by $\mathcal{O}_{U_\zeta}$ the structure sheaf of the basic wide open subdomain U_ζ. Let B be an open ball and S be an open annulus, with associated length $-\log(\rho)$, adjacent to ζ. Then $\mathcal{O}_{U_\zeta}(B)$ is given as all power series with radius of convergence 1, and $\mathcal{O}_{U_\zeta}(S)$ is given as all Laurent series that converge for $\rho < |T| < 1$.

Definition 5.1.21. Let X be a K-analytic space and $x \in X$. As in section 9.1, [Ber90], $s(x)$ is defined to be the transcendence degree of $\widetilde{\mathcal{H}(x)}$ over $k = \widetilde{K}$. Furthermore $t(x)$ is defined as the dimension of the $\mathbb{Q}$-vector space $\sqrt{|\mathcal{H}(x)^*|}/\sqrt{|K^*|}$. For the definition of the field $\mathcal{H}(x)$ see for Remark 1.2.2, [Ber90].
Eventually X_{st} is defined to be the set of all points $x \in X$ with $s(x) = t(x) = 0$. For a subset $V \subseteq X$, one sets $V_{\mathrm{st}} := V \cap X_{\mathrm{st}}$.

Remark 5.1.22. By the first paragraph of the introduction of [Ber07], if X is a smooth K-analytic space, $x \in X_{\mathrm{st}}$ is equivalent to the property that x admits a fundamental system of étale neighbourhoods isomorphic to an open ball. An étale neighbourhood of x is an open subset U with $x \in U$ such that $U \hookrightarrow X$ is an étale morphism.
This equivalence holds also for an open subset $V \subseteq X$ instead of the whole space X because, if $x \in V$ has a fundamental system of étale neighbourhoods isomorphic to an open ball in X, it has a fundamental system of étale neighbourhoods isomorphic to an open ball in V, too. Note that this, in general, does not hold for a non-open subset V.

Lemma 5.1.23. *Let $x \in \left(\mathbb{A}^1_K\right)^{\mathrm{an}}$. Berkovich classified in 1.4.4, [Ber90] the points of $\left(\mathbb{A}^1_K\right)^{\mathrm{an}}$ in four different types. For them the following properties hold:*

(i) If x is of type (1), then $s(x) = 0$ and $t(x) = 0$.
(ii) If x is of type (2), then $s(x) = 1$ and $t(x) = 0$.
(iii) If x is of type (3), then $s(x) = 0$ and $t(x) = 1$.
(iv) If x is of type (4), then $s(x) = 0$ and $t(x) = 0$.

Therefore just the points of type (1) and (4) are in $\left(\mathbb{A}_K^1\right)^{\mathrm{an}}_{\mathrm{st}}$*. These are exactly the end points of the Berkovich tree, which is the visualization of* $\left(\mathbb{A}_K^1\right)^{\mathrm{an}}$*. Note that, in harmony with Remark 5.1.22, the points of type (1) and (4) are the only ones that admit a fundamental system of étale neighbourhoods isomorphic to an open ball.*

Proof. Section 2.2, [Ber07]. □

Remark 5.1.24. Let X be a smooth K-analytic curve. By Lemma 2.1.15, for any $x \in X$ there exists an open neighbourhood $x \in U \subseteq X$ such that there exists an étale morphism

$$U \longrightarrow \left(\mathbb{A}_K^1\right)^{\mathrm{an}}.$$

The type of the image of x under this étale morphism is said to be the type of x. By the last paragraph before Proposition 2.2.1 in section 2.2, [Ber07], the type of the image of x does not depend on the choice of the étale morphism and is accordingly well-defined.

Lemma 5.1.25. *Let X be a smooth, proper, connected $\mathbb{C}_p$-curve. Then $X^{\mathrm{an}}_{\mathrm{st}}$ consists exactly of all the points of type (1) and (4) in X^{an}. These are the end points in the visualization of X^{an}.*

Proof. Let $x \in X^{\mathrm{an}}$.
Case 1: x is in an open ball B of the semistable decomposition.

Then

$$B \cong B(1)_+ \subseteq \left(\mathbb{A}_K^1\right)^{\mathrm{an}}$$

and one may apply Lemma 5.1.23. This means B_{st} is exactly the set of the points of type (1) and (4) in B.

Case 2: x is in an open annulus S of the semistable decomposition.

Apply Lemma 5.1.23 to

$$S \cong S(\rho)_+ \subseteq \left(\mathbb{A}_K^1\right)^{\mathrm{an}}$$

for some $\rho \in |\mathbb{C}_p^\times|$. As a consequence, S_{st} is exactly the set of the points of type (1) and (4) in S.

Case 3: x is a vertex in the semistable decomposition.

The vertex is by definition (preamble of section 1.2, [BPR13]) a point of type (2).

□

Remark 5.1.26. The equivalence of Remark 5.1.22 holds also for an underlying affinoid Y of a basic wide open subdomain, even though Y is generally not open. Y consists of one vertex ζ and open balls adjacent to ζ. Let X be the corresponding curve. The vertex ζ does not fulfill the requirements to be in Y_{st} because it is not in $X^{\mathrm{an}}_{\mathrm{st}}$ either. The remaining set $Y\setminus\{\zeta\}$ is an open subset, for which Remark 5.1.22 can be applied.

Definition 5.1.27. A **naive analytic function** f on an open subset $U \subseteq X^{\text{an}}$ is defined to be a map that allocates to every $x \in U_{\text{st}}$ a value $f(x) \in \mathcal{H}(x)$ such that there is an open neighbourhood $V \subseteq U_{\text{st}}$ of x with $f = g$ on V for a $g \in \mathcal{O}_U(V)$. The set of naive analytic functions on U is denoted by $\mathfrak{N}(U)$. Furthermore, the sheaf of $\mathcal{O}_{X^{\text{an}}}$-algebras is defined by the correspondence

$$U \longmapsto \mathfrak{N}(U)$$

and denoted by $\mathfrak{N}_{X^{\text{an}}}$.

Remark 5.1.28. It is convenient to call the functions of $\mathcal{O}_U(U)$ analytic functions on U. Then naive analytic functions are the Berkovich analogue of locally analytic functions.

Example 5.1.29. Any branch of the logarithm on an open annulus is a naive analytic function.

Definition 5.1.30. Let U be a connected open subset of X^{an} and M be a subset of $\mathfrak{N}(U)$. It is said that M satisfies the **uniqueness principle** on U if the following property holds:
If two elements in M coincide on a non-empty open subset of $U(\mathbb{C}_p)$, they coincide on the whole $U(\mathbb{C}_p)$.

Example 5.1.31. $\mathcal{O}_{X^{\text{an}}}(U)$ satisfies the uniqueness principle.

Definition 5.1.32. Let U be an open subset of X^{an} and define

$$A(U) := \mathcal{O}_{X^{\text{an}}}(U_{\text{st}}).$$

This definition can be extended to an underlying affinoid Y of a basic wide open subdomain U_ζ because $Y \backslash \{\zeta\}$ is a union of open balls (see for Remark 5.1.26).

Definition 5.1.33. Let U be an open subset of X^{an} and Log a fixed branch of the logarithm. Then it will be defined

$$A_{\text{Log}}(U) := A(U)\left[\{\text{Log}(f) : f \in A(U)^*\}\right].$$

Lemma 5.1.34. *Let U be an open annulus in X^{an}. Then*

$$A_{\text{Log}}(U) = A(U)\left[\text{Log}(z)\right].$$

Proof. Corollary 2.2.a, [Col82]. □

Remark 5.1.35. In Remark 5.1.13 it was written that it is already possible to formally antidifferentiate Laurent series on $\mathbb{C}_p^\times$ by using Log. For this reason, the set $A(U)$ was extended by Log and afterwards denoted by $A_{\text{Log}}(U)$. It would be nice if this bigger class of functions could be antidifferentiated, too. Indeed, this can be done.

Theorem 5.1.36. *Let U be an open annulus. Then it holds*

$$A_{\text{Log}}(U)\,\mathrm{d}z = \mathrm{d}A_{\text{Log}}(U).$$

Proof. "$\supseteq$" is clear.
"$\subseteq$": Let $\omega \in A_{\mathrm{Log}}(U)\,\mathrm{d}z$. Then it can be written as

$$\omega = \sum_{i=0}^{n} g_i(z) \operatorname{Log}^i(z)\,\mathrm{d}z$$

for an $n \in \mathbb{N}$ and $g_i \in A(U)$. The claim will be shown by induction:
For $n = 0$ it holds

$$\omega = g_0(z)\,\mathrm{d}z$$

and hence $\omega \in A(U)\,\mathrm{d}z$. If the coefficient of $\frac{\mathrm{d}z}{z}$ is zero, one can formally antidifferentiate g_0 and gets $\omega \in \mathrm{d}A(U)$. If not, the property

$$\frac{\mathrm{d}z}{z} = \mathrm{d}\operatorname{Log}(z)$$

yields at least $\omega \in \mathrm{d}A_{\mathrm{Log}}(U)$.
Suppose now that the claim is true for $n \in \mathbb{N}$.
Consider

$$\omega = \sum_{i=0}^{n+1} g_i(z) \operatorname{Log}^i(z)\,\mathrm{d}z.$$

Denote the coefficient of $\frac{1}{z}$ in $g_{n+1}(z)$ by c. Define

$$g'_{n+1}(z) := g_{n+1}(z) - c\tfrac{1}{z}.$$

Then one can formally antidifferentiate $g'_{n+1}(z)$ and gets a function $h \in A(U)$ with

$$\mathrm{d}h = g'_{n+1}(z)\,\mathrm{d}z.$$

This function allows to write

$$\begin{aligned}
&\mathrm{d}\Big(\tfrac{c}{n+2} \operatorname{Log}^{n+2}(z) + h \operatorname{Log}^{n+1}(z)\Big) - (n+1)h\tfrac{1}{z} \operatorname{Log}^n(z)\,\mathrm{d}z \\
&= c \operatorname{Log}^{n+1}(z)\tfrac{1}{z}\,\mathrm{d}z + \operatorname{Log}^{n+1}(z)\,\mathrm{d}h \\
&\quad + h \cdot (n+1) \operatorname{Log}^n(z)\tfrac{1}{z}\,\mathrm{d}z - (n+1)h\tfrac{1}{z} \operatorname{Log}^n(z)\,\mathrm{d}z \\
&= c \operatorname{Log}^{n+1}(z)\tfrac{1}{z}\,\mathrm{d}z + \operatorname{Log}^{n+1}(z) g'_{n+1}(z)\,\mathrm{d}z \\
&= c \operatorname{Log}^{n+1}(z)\tfrac{1}{z}\,\mathrm{d}z + \operatorname{Log}^{n+1}(z)\left(g_{n+1}(z) - c\tfrac{1}{z}\right)\,\mathrm{d}z \\
&= c \operatorname{Log}^{n+1}(z)\tfrac{1}{z}\,\mathrm{d}z + g_{n+1}(z) \operatorname{Log}^{n+1}(z)\,\mathrm{d}z - c\tfrac{1}{z} \operatorname{Log}^{n+1}(z)\,\mathrm{d}z \\
&= g_{n+1}(z) \operatorname{Log}^{n+1}(z)\,\mathrm{d}z
\end{aligned}$$

As the claim holds for $n \in \mathbb{N}$ and $(n+1)h\frac{1}{z}$ is in $A(U)$, one has

$$-(n+1)h\tfrac{1}{z} \operatorname{Log}^n(z)\,\mathrm{d}z \in \mathrm{d}A_{\mathrm{Log}}(U).$$

Since

$$\tfrac{c}{n+2}\operatorname{Log}^{n+2}(z) + h\operatorname{Log}^{n+1}(z) \in A_{\mathrm{Log}}(U),$$

it follows

$$\begin{aligned}&g_{n+1}(z)\operatorname{Log}^{n+1}(z)\ \mathrm{d}z\\ &= \mathrm{d}\Big(\tfrac{c}{n+2}\operatorname{Log}^{n+2}(z) + h\operatorname{Log}^{n+1}(z)\Big) - (n+1)h\tfrac{1}{z}\operatorname{Log}^{n}(z)\ \mathrm{d}z \in \mathrm{d}A_{\mathrm{Log}}(U).\end{aligned}$$

Hence

$$\omega \in \mathrm{d}A_{\mathrm{Log}}(U)$$

for $n+1$. This proves the theorem by induction. □

5.1.3 Logarithmic F-crystals

With section 5.1.2 one is now able to integrate not just analytic functions on open annuli, but also the bigger class of functions $A_{\mathrm{Log}}(U)$, which is a subset of the naive analytic functions on an open annulus U.
Still, the problem passing a vertex is not tackled. Doing this is the goal of this section. It will be continued with the same assumptions from before.

Definition 5.1.37. Suppose $M \subseteq \mathfrak{N}(U_\zeta)$ is an $A(U_\zeta)$-module. If there is an inclusion

$$i : W \hookrightarrow U_\zeta$$

of a $\mathbb{C}_p$-analytic space W into U_ζ, it will be defined

$$M(W) := i^* M$$

and

$$\Omega_M(W) := i^*\Omega_M,$$

where

$$\Omega_M = M \otimes_{A(U_\zeta)} \mathfrak{N}(U_\zeta)\ \mathrm{d}z.$$

For $W = Y$ or $W = S$, where Y is the underlying affinoid of the basic wide open subdomain U_ζ and S is an open annulus adjacent to ζ, it follows that

$$M(W) = M \otimes_{A(U_\zeta)} A(W)$$

and

$$\Omega_M(W) = M(W) \otimes_{A(W)} A(W)\ \mathrm{d}z.$$

Remark 5.1.38. Robert F. Coleman managed to construct a so called **logarithmic F-crystal**, which is an $A(U_\zeta)$-module $M \subseteq \mathfrak{N}(U_\zeta)$, such that $A(U_\zeta) \subseteq M$ and some other conditions hold. Its definition in rigid analysis is written in the beginning of section IV, [Col82].
A logarithmic F-crystal M has some nice properties, among others

(i) $\mathrm{d}M \subseteq \Omega_M$
(ii) $M(Y) \subseteq A(Y)$, $M(S) \subseteq A_{\mathrm{Log}}(S)$ for all open annuli S of the semistable decomposition that are in U_ζ.

Theorem 5.1.39. *Let U_ζ be a basic wide open subdomain of X^{an}. Then $A(U_\zeta)$ is a logarithmic F-crystal.*

Proof. Theorem 5.1, [Col82]. □

Theorem 5.1.40. *There exists a unique minimal logarithmic F-crystal M' on U_ζ such that $M \subseteq M'$ and*

$$\Omega_M \subseteq \mathrm{d}M'.$$

This means that M' delivers antiderivatives for all functions of M.

Proof. Theorem 4.3, [Col82]. □

Definition 5.1.41. Applying Theorem 5.1.40 to the logarithmic F-crystal $A(U_\zeta)$, it can be defined recursively

$$A^0(U_\zeta) := A(U_\zeta)$$

and

$$A^n(U_\zeta) := A^{n-1}(U_\zeta)'$$

for $n \in \mathbb{N}\backslash\{0\}$ where the prime denotes the unique logarithmic F-crystal from Theorem 5.1.40.
Furthermore we define

$$A^\infty(U_\zeta) := \bigcup_{n\in\mathbb{N}} A^n(U_\zeta).$$

Corollary 5.1.42. *It holds*

$$\Omega_{A^\infty(U_\zeta)}(U_\zeta) = \mathrm{d}A^\infty(U_\zeta).$$

Proof. Follows directly from Remark 5.1.38, property (i) and Theorem 5.1.40. □

Definition 5.1.43. A **filtered K-algebra** is defined as a commutative K-algebra L with unity such that there is a sequence of K-vector subspaces

$$L_0 \subseteq L_1 \subseteq L_2 \subseteq ...$$

with

(i) $L^i \cdot L^j \subseteq L^{i+j}$ and
(ii) $L = \bigcup_{i=0}^{\infty} L^i$.

Let X be a smooth K-analytic space. Then a **filtered $\mathcal{O}_X$-algebra** is defined as a sheaf of $\mathcal{O}_X$-algebras $\mathcal{A}$ with a sequence

$$\mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq ...$$

of $\mathcal{O}_X$-modules such that

(i) $\mathcal{A}^i \cdot \mathcal{A}^j \subseteq \mathcal{A}^{i+j}$ and
(ii) $\mathcal{A} = \varinjlim \mathcal{A}^i$.

Remark 5.1.44. $A^{\infty}(U_\zeta)$, which is a ring by Proposition 5.4, [Col82] and furthermore a filtered $A(U_\zeta)$-algebra, is a really huge set of functions that delivers antiderivatives for analytic functions on U_ζ, their antiderivatives and so on. Consequently one can find for any $\omega \in A^{n-1}(U_\zeta)\,\mathrm{d}z$ an antiderivative in $A^n(U_\zeta)$. But this may not be unique. In order to tackle this problem, there is the following theorem.

Theorem 5.1.45. *The set of functions $A^n(U_\zeta)$ satisfies the uniqueness principle on U_ζ.*

Proof. Since $U_\zeta \subseteq X^{\mathrm{an}}$ is connected, applying Theorem 5.7, [Col82] gives the result. □

Corollary 5.1.46. *For $\omega = f(z)\,\mathrm{d}z \in A^{n-1}(U_\zeta)dz$ and $P, Q \in U_\zeta(\mathbb{C}_p)$, the integral*

$$\int_P^Q \omega = \int_P^Q \mathrm{d}F = F(Q) - F(P)$$

is well-defined, where $F \in A^n(U_\zeta)$ is an antiderivative of $f \in A^{n-1}(U_\zeta)$.

Proof. By Theorem 5.1.39 there exists an antiderivative $F \in A^n(U_\zeta)$ of $f \in A^{n-1}(U_\zeta)$ which is not necessarily unique. Let $F_1, F_2 \in A^n(U_\zeta)$ be two possible antiderivatives of f, meaning $\mathrm{d}F_1 = \mathrm{d}F_2 = \omega$. Any $\mathbb{C}_p$-rational point on U_ζ lies in an open ball (not necessarily from the semistable decomposition), contained in U_ζ, where one can apply the Poincaré lemma. Write B_P for the open ball that contains P and B_Q for the open ball that contains Q. Applying the Poincaré lemma, one knows that the antiderivatives on B_P respectively B_Q are unique up to a constant in $\mathbb{C}_p$. Hence it holds

$$F_1 - F_2 = a$$

on B_P for an $a \in \mathbb{C}_p$. The uniqueness principle (Theorem 5.1.45) provides $F_1 = F_2 + a$ on the $\mathbb{C}_p$-rational points of the whole basic wide open subdomain $U_\zeta(\mathbb{C}_p)$ and especially on $B_Q(\mathbb{C}_p)$. Since

$$\begin{aligned}\int_P^Q \mathrm{d}F_1 &= F_1(Q) - F_1(P) \\ &= (F_2(Q) + a) - (F_2(P) + a) \\ &= F_2(Q) - F_2(P) \\ &= \int_P^Q \mathrm{d}F_2,\end{aligned}$$

the integral $\int_P^Q \omega$ is well-defined for $\omega \in A^{n-1}(U_\zeta)\,\mathrm{d}z$. □

Corollary 5.1.47. *For $\omega = f(z)\,\mathrm{d}z \in A^\infty(U_\zeta)\,\mathrm{d}z$ and $P, Q \in U_\zeta(\mathbb{C}_p)$, the integral*

$$\int_P^Q \omega = \int_P^Q \mathrm{d}F = F(Q) - F(P)$$

is well-defined, where $F \in A^\infty(U_\zeta)$ is an antiderivative of $f \in A^\infty(U_\zeta)$.

Proof. Follows directly from Corollary 5.1.46 for an $n \in \mathbb{N}$ big enough. □

5.1.4 Historical integral by Coleman

After section 5.1.3 it is now possible to pass a vertex. But still, the integral is just defined within a basic wide open subdomain. The goal of this section is to extend the definition to the whole space X^{an}. Continue with the same assumptions as before.

Definition 5.1.48. Let Z be a topological space and

$$\gamma : [a, b] \longrightarrow Z$$

be a path on Z. A **subdivision** of the path γ is defined to be the set of a finite number $n \in \mathbb{N}\backslash\{0\}$ of paths

$$\gamma_i : [a_i, b_i] \longrightarrow Z$$

with $i \in \{1, ..., n\}$ such that

(i) $\gamma_i(b_i) = \gamma_{i+1}(a_{i+1})$ for $i \in \{1, ..., n-1\}$,
(ii) $\gamma_1(a_1) = \gamma(a)$ and $\gamma_n(b_n) = \gamma(b)$ and
(iii) the concatenation $\gamma_1 * \gamma_2 * ... * \gamma_n$ is equal to γ.

$a \leq b$ and $a_i \leq b_i$ for $i \in \{1, ..., n\}$ are real numbers.

Lemma 5.1.49. *Let X be a smooth, proper, connected $\mathbb{C}_p$-curve, along with a semistable model $\mathcal{X}$, and V be the vertex set of the semistable decomposition of X^{an}. Furthermore,*

$$\gamma : [0, 1] \longrightarrow X^{\mathrm{an}}$$

is a path from a $\mathbb{C}_p$-rational point P to a $\mathbb{C}_p$-rational point Q.
Then there exists a subdivision of γ into $n \in \mathbb{N}\backslash\{0\}$ paths

$$\gamma_i : [0, 1] \longrightarrow X^{\mathrm{an}}$$

with the property that for any $i \in \{1, ..., n\}$ there exists a vertex $\zeta_i \in V$ such that

$$\mathrm{Im}(\gamma_i) \subseteq U_{\zeta_i}.$$

Proof. Since any basic wide open subdomain $U_\zeta \subseteq X^{\mathrm{an}}$ is open, the pre-image

$$\gamma^{-1}(U_\zeta) \subseteq [0,1]$$

under the continuous function γ is an open subset, too. The analytification X^{an} of the curve may be covered by the basic wide open subdomains U_ζ with $\zeta \in V$. Therefore, $[0,1]$ may be covered by the pre-images $\gamma^{-1}(U_\zeta)$ with $\zeta \in V$.
As $\gamma^{-1}(U_\zeta) \subseteq [0,1]$ is open, the union

$$(-\infty, 0) \cup \gamma^{-1}(U_\zeta) \cup (1, +\infty)$$

is an open subset of $\mathbb{R}$. A subset of $\mathbb{R}$ is open if and only if it is the union of countably many open intervals in $\mathbb{R}$. Since $(-\infty, 0) \cup \gamma^{-1}(U_\zeta) \cup (1, +\infty)$ is open in $\mathbb{R}$, this fact can be applied to it. By restricting this set to $[0,1]$, the set $\gamma^{-1}(U_\zeta)$ is the union of countably many open intervals in $[0,1]$. This means that the pre-image $\gamma^{-1}(U_\zeta) \subseteq [0,1]$ can be written as

$$\gamma^{-1}(U_\zeta) = \bigcup_{j \in \mathbb{N}} I_{j,\zeta}$$

where $I_{j,\zeta}$ is an open interval in $[0,1]$.
From the first and second paragraph it now follows that

$$[0,1] = \bigcup_{\zeta \in V} \gamma^{-1}(U_\zeta) = \bigcup_{\zeta \in V} \bigcup_{j \in \mathbb{N}} I_{j,\zeta},$$

which means that $\bigcup_{\zeta \in V} \bigcup_{j \in \mathbb{N}} I_{j,\zeta}$ is an open cover of $[0,1]$. The interval $[0,1]$ is compact because it is closed and bounded. Therefore one can choose a finite subset of the open intervals

$$\{I_{j,\zeta}\}_{j \in \mathbb{N}, \zeta \in V}$$

with the result that its union still covers $[0,1]$. To sum up, we have a finite number of open intervals $I_{j,\zeta} \subseteq [0,1]$ that still covers $[0,1]$.
With that said, it is possible to choose a finite number of real numbers

$$0 = r_0 < r_1 < r_2 < ... < r_{n-1} < r_n = 1$$

with $n \in \mathbb{N}\backslash\{0\}$, such that for any $i \in \{1, ..., n\}$ there exists an open interval $I_{j,\zeta} \subseteq [0,1]$ with

$$[r_{i-1}, r_i] \subseteq I_{j,\zeta}.$$

This yields

$$\gamma([r_{i-1}, r_i]) \subseteq \gamma(I_{j,\zeta}) \subseteq U_\zeta. \tag{5.10}$$

By defining

$$\begin{aligned} \gamma_i : [r_{i-1}, r_i] &\longrightarrow X^{\mathrm{an}}, \\ r &\longmapsto \gamma(r) \end{aligned}$$

one gets continuous maps γ_i fulfilling the conditions of Definition 5.1.48. The property $\mathrm{Im}(\gamma_i) \subseteq U_\zeta$ was shown in (5.10). Finally, by re-parametrization, one gets the searched-for subdivision of γ. $\square$

Corollary 5.1.50. *Let X be a smooth, proper, connected $\mathbb{C}_p$-curve, along with a semistable model $\mathcal{X}$, and V be the vertex set of the semistable decomposition of X^{an}. Furthermore,*

$$\gamma : [0,1] \longrightarrow X^{\mathrm{an}}$$

is a path from a $\mathbb{C}_p$-rational point P to a $\mathbb{C}_p$-rational point Q.
Then there exist an $n \in \mathbb{N}\backslash\{0\}$ and paths

$$\gamma_i : [0,1] \longrightarrow X^{\mathrm{an}}$$

with $i \in \{1, ..., n\}$ fulfilling the following properties:

(i) For any $i \in \{1, ..., n\}$ there exists a vertex $\zeta_i \in V$ such that

$$\mathrm{Im}(\gamma_i) \subseteq U_{\zeta_i}.$$

(ii) The path γ is homotopy equivalent to the concatenation $\gamma_1 * \gamma_2 * ... * \gamma_n$.
(iii) The end points of the γ_i are $\mathbb{C}_p$-rational.

Proof. At first one may assume $\zeta_i \neq \zeta_{i+1}$ for $i \in \{1, ..., n-1\}$ in Lemma 5.1.49 for the following reason: If there exists a $j \in \{1, ..., n-1\}$ with $\zeta_j = \zeta_{j+1}$, one can concatenate γ_j and γ_{j+1} and redefine

$$\gamma_i' := \begin{cases} \gamma_i & \text{for } i < j \\ \gamma_i * \gamma_{i+1} & \text{for } i = j \\ \gamma_{i+1} & \text{for } i > j \end{cases}$$

for $i \in \{1, ..., n-1\}$. This new subdivision also fulfils the property of Lemma 5.1.49 as

$$\begin{aligned} \mathrm{Im}(\gamma_j') &= \mathrm{Im}(\gamma_j * \gamma_{j+1}) \\ &= \mathrm{Im}(\gamma_j) \cup \mathrm{Im}(\gamma_{j+1}) \\ &\subseteq U_{\zeta_j} \cup U_{\zeta_{j+1}} \\ &= U_{\zeta_j}. \end{aligned}$$

Let now γ_i be a subdivision fulfilling the property of Lemma 5.1.49 and the assumption from above. Then, for $i \in \{1, ..., n-1\}$, the point

$$P_i := \gamma_i(1) = \gamma_{i+1}(0)$$

lies in the intersection of the two basic wide open subdomains U_{ζ_i} and $U_{\zeta_{i+1}}$ because

$$\gamma_i(1) \in \mathrm{Im}(\gamma_i) \subseteq U_{\zeta_i}$$

and

$$\gamma_{i+1}(0) \in \mathrm{Im}(\gamma_{i+1}) \subseteq U_{\zeta_{i+1}}.$$

A non-empty intersection of two different basic wide open subdomains is the disjoint union of open annuli from the semistable decomposition. This means that the point P_i lies in an open annulus

$$A_i \subseteq U_{\zeta_i} \cap U_{\zeta_{i+1}}$$

of the semistable decomposition. Since any open annulus of the semistable decomposition is isomorphic to $S(\rho)_+$, with $\rho \in |\mathbb{C}_p^\times|$, it contains at least one $\mathbb{C}_p$-rational point P_i'. Since A_i is path-connected, there exist paths

$$\gamma_{i,\mathrm{go}} : [0,1] \longrightarrow A_i$$

and

$$\gamma_{i,\mathrm{return}} : [0,1] \longrightarrow A_i$$

for $i \in \{1, ..., n-1\}$, such that

$$\gamma_{i,\mathrm{go}}(0) = P_i = \gamma_{i,\mathrm{return}}(1)$$

and

$$\gamma_{i,\mathrm{go}}(1) = P_i' = \gamma_{i,\mathrm{return}}(0).$$

The open annulus A_i is furthermore simply-connected and therefore the cycle

$$\gamma_{i,\mathrm{go}} * \gamma_{i,\mathrm{return}}$$

is homotopy equivalent to 0.
The definition

$$\gamma_i' := \begin{cases} \gamma_i * \gamma_{i,\mathrm{go}} & \text{for } i = 1 \\ \gamma_{i-1,\mathrm{return}} * \gamma_i * \gamma_{i,\mathrm{go}} & \text{for } 1 < i < n \\ \gamma_{i-1,\mathrm{return}} * \gamma_i & \text{for } i = n. \end{cases}$$

fulfils the required properties:

(i) For $1 < i < n$ it holds

$$\begin{aligned}\mathrm{Im}(\gamma_i') &= \mathrm{Im}(\gamma_{i-1,\mathrm{return}} * \gamma_i * \gamma_{i,\mathrm{go}}) \\ &= \mathrm{Im}(\gamma_{i-1,\mathrm{return}}) \cup \mathrm{Im}(\gamma_i) \cup \mathrm{Im}(\gamma_{i,\mathrm{go}}) \\ &\subseteq A_{i-1} \cup U_{\zeta_i} \cup A_i \\ &= U_{\zeta_i}.\end{aligned}$$

The cases $i = 1$ and $i = n$ follow by setting $\gamma_{0,\mathrm{return}} := 0$ and $\gamma_{n,\mathrm{go}} := 0$.

(ii) Using the equality sign for homotopy equivalence, it follows that

$$\begin{aligned}&\gamma_1' * \gamma_2' * \dots * \gamma_n' \\ &= (\gamma_1 * \gamma_{1,\mathrm{go}}) * (\gamma_{1,\mathrm{return}} * \gamma_2 * \gamma_{2,\mathrm{go}}) * \dots * (\gamma_{n-1,\mathrm{return}} * \gamma_n) \\ &= \gamma_1 * \underbrace{\gamma_{1,\mathrm{go}} * \gamma_{1,\mathrm{return}}}_{=0} * \gamma_2 * \gamma_{2,\mathrm{go}} * \dots * \gamma_{n-1,\mathrm{return}} * \gamma_n \\ &= \gamma_1 * \gamma_2 * \dots * \gamma_n \\ &= \gamma.\end{aligned}$$

(iii) The end points of the paths γ_i' are P, Q and P_i' for $i \in \{1, \dots, n-1\}$. These are all $\mathbb{C}_p$-rational.

□

Definition 5.1.51. Let X be a smooth, proper, connected $\mathbb{C}_p$-curve, along with a semistable model $\mathcal{X}$, and $\omega \in \Omega^1_{X/\mathbb{C}_p}$ be a differential one-form on X. Furthermore

$$\gamma : [0,1] \longrightarrow X^{\mathrm{an}}$$

denotes a path from a $\mathbb{C}_p$-rational point P to a $\mathbb{C}_p$-rational point Q.
By Corollary 5.1.50 there exist paths

$$\gamma_i : [0,1] \longrightarrow X^{\mathrm{an}}$$

with $i \in \{1, \dots, n\}$ fulfilling the conditions listed in Corollary 5.1.50. Define

$$R_i := \gamma_i(1) = \gamma_{i+1}(0)$$

for $i \in \{1, \dots, n-1\}$, $R_0 := \gamma_0(0) = P$ and $R_n := \gamma_n(1) = Q$.

Then the **historical integral by Coleman** is defined as

$$\int_\gamma \omega := \sum_{i=1}^{n} \int_{R_{i-1}}^{R_i} \omega$$

where $\int_{R_{i-1}}^{R_i} \omega$ is well-defined by Corollary 5.1.47.

Theorem 5.1.52. *The historical integral of Coleman is well-defined.*

Proof. The case $n = 1$ is trivial. In the following it will be assumed $n > 1$.
For the proof of the theorem one has to show that the integral is independent from the choice of the γ_i. Firstly show that, if γ is a null-homotopic path, for any set of paths

$$\gamma_i : [0,1] \longrightarrow X^{\text{an}}$$

with $i \in \{1,...,n\}, n \in \mathbb{N}\backslash\{0\}$ fulfilling the conditions of Corollary 5.1.50 it holds

$$\int_\gamma \omega = \sum_{i=1}^{n} \int_{R_{i-1}}^{R_i} \omega = 0.$$

Let γ be a null-homotopic path and the γ_i as above. Then one can assume that in any open annulus of the semistable decomposition there do not lie two or more different starting respectively end points of the γ_i: Let A be an open annulus of the semistable decomposition and P, P' are two different starting respectively end points of the γ_i in A. As in the proof of Corollary 5.1.50, there exist paths

$$\gamma_{\text{go}} : [0,1] \longrightarrow A$$

and

$$\gamma_{\text{return}} : [0,1] \longrightarrow A$$

such that

$$\gamma_{\text{go}}(0) = P = \gamma_{\text{return}}(1) \tag{5.11}$$

and

$$\gamma_{\text{go}}(1) = P' = \gamma_{\text{return}}(0). \tag{5.12}$$

Since

$$\gamma_{i,\text{go}} * \gamma_{i,\text{return}}$$

is homotopy equivalent to 0, one can transform any path γ_i with P as an end point into $\gamma_i * \gamma_{\text{go}}$ and any path γ_{i+1} (or γ_1 if $i = n$) with P as a starting point into $\gamma_{\text{return}} * \gamma_{i+1}$. Note that the concatenation of the γ_i is null-homotopic and therefore a cycle. This modification does not change the result of the integral:
On an open annulus of the semistable decomposition any $\omega \in \Omega^1_{X/\mathbb{C}_p}$ has an antiderivative. Finally, by Corollary 5.1.47, the integration of ω along γ_{go} is defined as

$$\begin{aligned}\int_{\gamma_{\text{go}}(0)}^{\gamma_{\text{go}}(1)} \omega &= \int_{\gamma_{\text{go}}(0)}^{\gamma_{\text{go}}(1)} \mathrm{d}F_{\text{go}} \\ &= F_{\text{go}}(\gamma_{\text{go}}(1)) - F_{\text{go}}(\gamma_{\text{go}}(0))\end{aligned}$$

and the integration of ω along γ_{return} is defined as

$$\begin{aligned}\int_{\gamma_{\text{return}}(0)}^{\gamma_{\text{return}}(1)} \omega &= \int_{\gamma_{\text{return}}(0)}^{\gamma_{\text{return}}(1)} \mathrm{d}F_{\text{return}} \\ &= F_{\text{return}}(\gamma_{\text{return}}(1)) - F_{\text{return}}(\gamma_{\text{return}}(0))\end{aligned}$$

where F_{go} and F_{return} are elements of $A_{\text{Log}}(A)$ with $\omega = \mathrm{d}F_{\text{go}}$ and $\omega = \mathrm{d}F_{\text{return}}$. Since the antiderivatives differ just by a constant $c \in \mathbb{C}_p$, one has

$$F_{\text{go}} = F_{\text{return}} + c$$

on A. This yields

$$\begin{aligned}&\int_{\gamma_{\text{go}}(0)}^{\gamma_{\text{go}}(1)} \omega + \int_{\gamma_{\text{return}}(0)}^{\gamma_{\text{return}}(1)} \omega \\ &= \int_{\gamma_{\text{go}}(0)}^{\gamma_{\text{go}}(1)} \mathrm{d}F_{\text{go}} + \int_{\gamma_{\text{return}}(0)}^{\gamma_{\text{return}}(1)} \mathrm{d}F_{\text{return}} \\ &= F_{\text{go}}(\gamma_{\text{go}}(1)) - F_{\text{go}}(\gamma_{\text{go}}(0)) + F_{\text{return}}(\gamma_{\text{return}}(1)) - F_{\text{return}}(\gamma_{\text{return}}(0)) \\ &= (F_{\text{return}} + c)(\gamma_{\text{go}}(1)) - (F_{\text{return}} + c)(\gamma_{\text{go}}(0)) + F_{\text{return}}(\gamma_{\text{return}}(1)) - F_{\text{return}}(\gamma_{\text{return}}(0)) \\ &= F_{\text{return}}(\gamma_{\text{go}}(1)) - F_{\text{return}}(\gamma_{\text{go}}(0)) + F_{\text{return}}(\gamma_{\text{return}}(1)) - F_{\text{return}}(\gamma_{\text{return}}(0)) \\ &= F_{\text{return}}(\gamma_{\text{go}}(1)) - F_{\text{return}}(\gamma_{\text{go}}(0)) + F_{\text{return}}(\gamma_{\text{return}}(1)) - F_{\text{return}}(\gamma_{\text{return}}(0)) \\ &\overset{(5.11 \text{ and } (5.12)}{=} 0.\end{aligned}$$

Therefore one can modify the γ_i, such that in any annulus there do not lie two or more different starting respectively end points, without changing the result of the integral. Furthermore, we may assume that, if the semistable decomposition contains open annuli (if not, there exists just one basic wide open subdomain and hence the historical integral by Coleman is trivial), every starting and end point lies in an open annulus of the semistable decomposition for the following reason: By the proof of Corollary 5.1.50 it is already possible to assume $\zeta_i \neq \zeta_{i+1}$ for $i \in \{1, ..., n-1\}$ and therefore

$$\gamma_i(1) = \gamma_{i+1}(0) \in A_i \subseteq U_{\zeta_i} \cap U_{\zeta_{i+1}} \tag{5.13}$$

for $i \in \{1, ..., n-1\}$, where A_i is an open annulus of the semistable decomposition. Since $\gamma_1 * \gamma_2 * ... * \gamma_n$ is now a cycle, property (5.13) can be extended to

$$\gamma_n(1) = \gamma_1(0).$$

An interim conclusion is the following: If $n > 1$ and γ is null-homotopic, we may assume the following properties for the paths γ_i with $i \in \{1, ..., n\}$ in addition to Corollary 5.1.50:

(i) $\zeta_i \neq \zeta_{i+1}$ for $i \in \{1, ..., n-1\}$ and $\zeta_n \neq \zeta_1$.
(ii) Any ending of γ_i is $\mathbb{C}_p$-rational and lies in an open annulus of the semistable decomposition.

(iii) Within one open annulus of the semistable decomposition there do not lie two or more different endings of the γ_i.

Let now γ be a null-homotopic path and the γ_i as above. As any γ_i is contained in a single basic wide open subdomain, there does not exist a γ_i that passes two different vertices. Let A be an open annulus of the semistable decomposition and ζ_1 and ζ_2 two vertices adjacent to A. On A there is just one possible starting or end point, denoted by P_A. This means that, whenever $\gamma_1 * \gamma_2 * ... * \gamma_n$ passes from ζ_1 to ζ_2, it has to pass the point P_A.
Let $P_A \in A$ be an end point of a γ_i, and therefore starting point of γ_{i+1} (or γ_1 if $i = n$). Then the vertices ζ_i and ζ_{i+1}, which are passed by γ_i and γ_{i+1} respectively, are different, as shown in the beginning of the proof of Corollary 5.1.50. Hence the path $\gamma_1 * \gamma_2 * ... * \gamma_n$ passes from ζ_i to ζ_{i+1}, which are the two vertices adjacent to P_A. This means that, whenever $\gamma_1 * \gamma_2 * ... * \gamma_n$ passes P_A, the previous vertex and the next vertex on $\gamma_1 * \gamma_2 * ... * \gamma_n$ are the two vertices adjacent to A.
Since γ is a null-homotopic path, $\gamma_1 * \gamma_2 * ... * \gamma_n$ is also null-homotopic. Since all the open balls of the analytification are simply-connected, one can retract $\gamma_1 * \gamma_2 * ... * \gamma_n$ to the skeleton, which yields a null-homotopic path on the skeleton. Together with the previous two paragraphs, this means that any edge between two vertices is passed by $\gamma_1 * \gamma_2 * ... * \gamma_n$ in one direction as often as in the reverse direction. Hence, if we drop the retraction and go back to X^{an}, $\gamma_1 * \gamma_2 * ... * \gamma_n$ goes from a vertex ζ_1 to a vertex ζ_2 as often as from ζ_2 to ζ_1.
As a result of the previous three paragraphs, for an open annulus A of the semistable decomposition and a vertex ζ adjacent to A, the numbers

$$n_{\zeta,P_A} := \text{number of paths } \gamma_i \text{ passing } \zeta \text{ and having } P_A \text{ as end point}$$

and

$$n_{P_A,\zeta} := \text{number of paths } \gamma_i \text{ passing } \zeta \text{ and having } P_A \text{ as starting point}$$

coincide.
Consider a path γ_i on $U_{\zeta_i} \subseteq X^{\mathrm{an}}$. Any differential one-form $\omega \in \Omega^1_{X/\mathbb{C}_p}$, restricted to the basic wide open subdomain U_{ζ_i}, is in $A^\infty(U_{\zeta_i})\,\mathrm{d}z$, which means $\omega = f\,\mathrm{d}z$ for an $f \in A^\infty(U_{\zeta_i})$. Furthermore, by Corollary 5.1.42, there exists an antiderivative $F \in A^\infty(U_{\zeta_i})$ for $f \in A^\infty(U_{\zeta_i})$, which means $\omega = \mathrm{d}F$. Finally, by Corollary 5.1.47, the integration of ω along γ_i is defined as

$$\int_{\gamma_i(0)}^{\gamma_i(1)} \omega = \int_{\gamma_i(0)}^{\gamma_i(1)} \mathrm{d}F = F(\gamma_i(1)) - F(\gamma_i(0)).$$

The antiderivative F is just defined up to a constant. Since this constant does not change the result, one can fix an antiderivative F_ζ of ω for any basic wide open subdomain U_ζ. For an open annulus A of the semistable decomposition, we denote the two vertices adjacent to A by ζ'_A and ζ''_A. We can assume that any starting and end point lies in such an A as shown before. Then it holds

$$\begin{aligned}
\int_\gamma \omega &= \sum_{i=1}^{n} \int_{R_{i-1}}^{R_i} \omega \\
&= \sum_{i=1}^{n} \left(F_{\zeta_i}(\gamma_i(1)) - F_{\zeta_i}(\gamma_i(0))\right) \\
&= \sum_{A} \Big(n_{\zeta'_A, P_A} F_{\zeta'_A}(P_A) - n_{P_A, \zeta'_A} F_{\zeta'_A}(P_A) + n_{\zeta''_A, P_A} F_{\zeta''_A}(P_A) - n_{P_A, \zeta''_A} F_{\zeta''_A}(P_A) \Big) \\
&= \sum_{A} \Big((n_{\zeta'_A, P_A} - n_{P_A, \zeta'_A}) F_{\zeta'_A}(P_A) + (n_{\zeta''_A, P_A} - n_{P_A, \zeta''_A}) F_{\zeta''_A}(P_A) \Big) \\
&= 0,
\end{aligned}$$

where A runs over all open annuli of the semistable decomposition. The penultimate line is 0 because

$$n_{\zeta_A, P_A} = n_{P_A, \zeta_A}$$

holds for any vertex ζ_A adjacent to A as mentioned before.
It has been proven that, for a null-homotopic path γ, the historical integral by Coleman is 0. Let γ and γ' be paths that are homotopy equivalent. Define the negative $-\gamma$ of a path γ to be the path defined by

$$\begin{aligned}
-\gamma : [0,1] &\longrightarrow X^{\mathrm{an}} \\
t &\longmapsto \gamma(1-t).
\end{aligned}$$

By Corollary 5.1.50, there exist $n, n' \in \mathbb{N}\backslash\{0\}$ and paths

$$\gamma_i : [0,1] \longrightarrow X^{\mathrm{an}}$$

and

$$\gamma'_j : [0,1] \longrightarrow X^{\mathrm{an}},$$

with $i \in \{1, ..., n\}$ and $j \in \{1, ..., n'\}$, fulfilling the conditions of Corollary 5.1.50 for γ and γ'. Then the paths

$$\gamma''_i : [0,1] \longrightarrow X^{\mathrm{an}}$$

with $i \in \{1, ..., n+n'\}$ and

$$\gamma''_i := \begin{cases} \gamma_i & \text{for } i \leq n \\ -\gamma'_{n+n'+1-i} & \text{for } i > n \end{cases}$$

fulfil the conditions of Corollary 5.1.50 for the concatenation $\gamma * (-\gamma')$. Therefore

$$\int_{\gamma''} \omega = \int_\gamma \omega + \int_{-\gamma'} \omega.$$

Since γ and γ' are homotopy equivalent, the concatenation $\gamma * (-\gamma')$ is null-homotopic. Hence

$$\begin{aligned} 0 &= \int_{\gamma''} \omega \\ &= \int_{\gamma} \omega + \int_{-\gamma'} \omega \\ &= \int_{\gamma} \omega - \int_{\gamma'} \omega \end{aligned}$$

and consequently

$$\int_{\gamma} \omega = \int_{\gamma'} \omega.$$

This means that the historical integral of Coleman is well-defined. □

Theorem 5.1.53. *Let X be a smooth, proper, connected $\mathbb{C}_p$-curve, along with a semistable model $\mathcal{X}$, let $\omega \in \Omega^1_{X/\mathbb{C}_p}$ be a differential one-form on X and let $\mathcal{P}(X^{\mathrm{an}})$ be the set of paths $\gamma : [0,1] \longrightarrow X^{\mathrm{an}}$ with starting and end points in $X(\mathbb{C}_p)$.*
Then the historical integral by Coleman is an integration theory in the sense of Definition 3.2.1, meaning it fulfils the following properties:

(i) If $U \subseteq X^{\mathrm{an}}$ is an open subdomain isomorphic to an open ball, and $\omega = \mathrm{d}f$ with f analytic on U, then

$$\int_{\gamma} \omega = f(\gamma(1)) - f(\gamma(0))$$

for all $\gamma : [0,1] \longrightarrow U$.

(ii) $\int_{\gamma} \omega$ only depends on the fixed end point homotopy class of γ.

(iii) If $\gamma_1, \gamma_2 \in \mathcal{P}(X^{\mathrm{an}})$ and $\gamma_2(0) = \gamma_1(1)$, then

$$\int_{\gamma_1 * \gamma_2} \omega = \int_{\gamma_1} \omega + \int_{\gamma_2} \omega,$$

where $\gamma_1 * \gamma_2$ is the concatenation of the two paths.

(iv) $\omega \longmapsto \int_{\gamma} \omega$ is linear in ω for a fixed γ.

Note that for a smooth, proper, connected $\mathbb{C}_p$-curve X it holds $\Omega^1_{X/\mathbb{C}_p} = Z^1_{\mathrm{dR}}(X)$.

Proof. (i) Let $U \subseteq X^{\mathrm{an}}$ be an open subdomain isomorphic to an open ball, and $\omega = \mathrm{d}f$ with f analytic on U. Then U is contained in an open ball B or open annulus S of the semistable decomposition. There always exists an antiderivative $F \in A(B)$ respectively $F \in A_{\mathrm{Log}}(S)$ with $\omega = \mathrm{d}F$ on U_ζ. Since the antiderivative f of ω on

U is unique up to a constant, there exists a constant $c \in \mathbb{C}_p$ with $F = f + c$ on U. Then Corollary 5.1.47 yields

$$\begin{aligned}\int_\gamma \omega &= \int_\gamma \mathrm{d}F \\ &= F(\gamma(1)) - F(\gamma(0)) \\ &= (f+c)(\gamma(1)) - (f+c)(\gamma(0)) \\ &= f(\gamma(1)) - f(\gamma(0))\end{aligned}$$

for all $\gamma : [0,1] \longrightarrow U$.

(ii) Let $\gamma, \gamma' \in \mathcal{P}(X^{\mathrm{an}})$ be two paths that are homotopy equivalent and let

$$\gamma_i : [0,1] \longrightarrow X^{\mathrm{an}}$$

with $i \in \{1, ..., n\}$ be paths fulfilling the conditions of Corollary 5.1.50. Since γ and γ' are homotopy equivalent, the concatenation $\gamma_1 * \gamma_2 * ... * \gamma_n$ is also homotopy equivalent to γ'. Therefore the paths γ_i also fulfil the conditions of Corollary 5.1.50 for γ'. Finally, with the same paths γ_i for both paths, the historical integral by Coleman provides the same result for γ and γ'.

(iii) For γ_1 and γ_2 exist paths as in Corollary 5.1.50. The union of these paths fulfils the conditions of Corollary 5.1.50 for the concatenation $\gamma_1 * \gamma_2$. This proves

$$\int_{\gamma_1 * \gamma_2} \omega = \int_{\gamma_1} \omega + \int_{\gamma_2} \omega.$$

(iv) The integral

$$\int_P^Q \omega = \int_P^Q \mathrm{d}F = F(Q) - F(P)$$

on a basic wide open subdomain is linear in F. Since ω, restricted to the basic wide open subdomain U_ζ, is in $A^\infty(U_\zeta)\,\mathrm{d}z$, linearity also holds for ω. As the historical integral by Coleman is just a sum of the integrals on basic wide open subdomains,

$$\omega \longmapsto \int_\gamma \omega$$

is linear in ω for a fixed γ.

□

Example 5.1.54. Take the smooth, proper, connected $\mathbb{C}_p$-curve X from Figure 5.1. Then its analytification is drawn in Figure 5.8. One has a differential one-form $\omega \in \Omega^1_{X/\mathbb{C}_p}$ and a path γ from the point $P \in X(\mathbb{C}_p)$ to the point $Q \in X(\mathbb{C}_p)$, sketched in the following. But, other than in Figure 5.5, this path does not lie in one basic wide open subdomain.

The solution is to divide the path in small parts, each of which lies in a basic wide open subdomain. In the case of Figure 5.8, this can be done by fixing a point $R \in X(\mathbb{C}_p)$

and splitting γ at this point. Then the path γ_1 from P to R is in the basic wide open subdomain U_{ζ_1} and the path γ_2 from R to Q is in the basic wide open subdomain U_{ζ_2}. The paths γ_1 and γ_2 fulfil the conditions of Corollary 5.1.50.

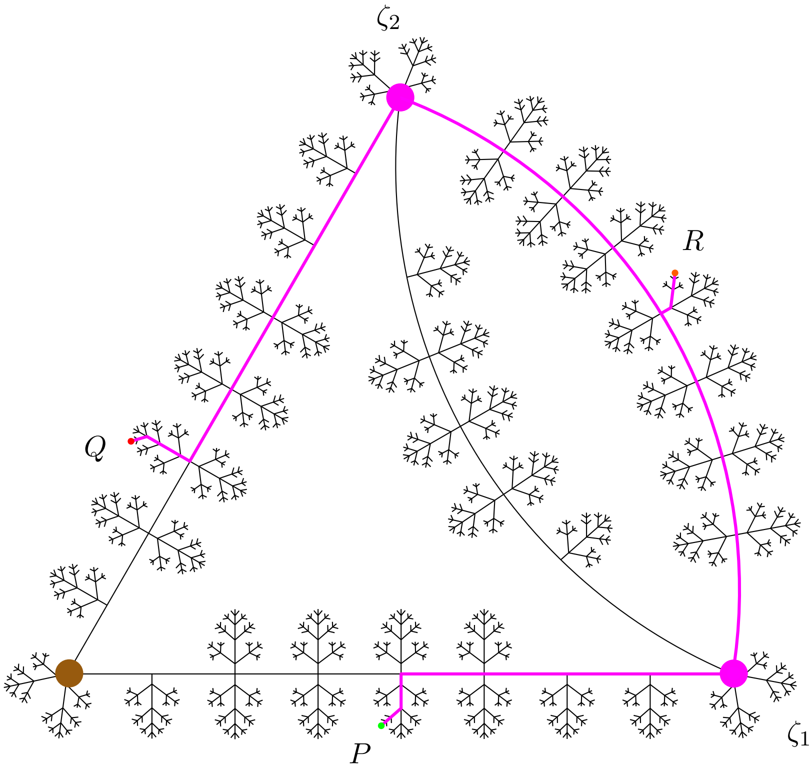


Figure 5.8: Integration along a path, passing the vertices ζ_1 and ζ_2

For these smaller paths it is possible to apply the techniques developed in this section. As the restriction of ω to the basic wide open subdomain U_{ζ_1} is in $A^\infty(U_{\zeta_1})\,\mathrm{d}z$, Corollary 5.1.47 delivers an $F_1 \in A^\infty(U_{\zeta_1})$ such that

$$\int_{\gamma_1} \omega = \int_P^R \omega = \int_P^R \mathrm{d}F_1 = F_1(R) - F_1(P).$$

As the same holds for U_{ζ_2}, there exists an $F_2 \in A^\infty(U_{\zeta_2})$ such that

$$\int_{\gamma_2} \omega = \int_R^Q \omega = \int_R^Q \mathrm{d}F_2 = F_2(Q) - F_2(R).$$

For the historical integral by Coleman it follows that

$$\begin{aligned}\int_\gamma \omega &= \int_P^R \omega + \int_R^Q \omega \\ &= \int_P^R \mathrm{d}F_1 + \int_R^Q \mathrm{d}F_2 \\ &= F_1(R) - F_1(P) + F_2(Q) - F_2(R) \\ &= F_2(Q) - F_1(P) + F_1(R) - F_2(R).\end{aligned}$$

As already shown in Theorem 5.1.52, this result is independent of the choice of the point R.

5.2 Modern approach by Berkovich

Vladimir G. Berkovich took the idea of Robert F. Coleman and extended it to any smooth $\mathbb{C}_p$-analytic space. Hence one is not restricted to curves anymore.
At first a definition of what the Berkovich-Coleman integration theory should be will be given, and afterwards it will be focused on the question whether it actually exists and if it is unique. Berkovich proved this in his book *Integration of One-forms on P-adic analytic spaces*, [Ber07]. Since the book extends to 156 pages with lots of special terms, in the following it will be restricted to just giving the cornerstones of its proof.

Definition 5.2.1. The **Berkovich-Coleman integration theory** is an integration theory

$$^{\mathrm{BC}}\!\!\int : \mathcal{P}(X) \times Z^1_{\mathrm{dR}}(X) \longrightarrow \mathbb{C}_p$$

for every smooth $\mathbb{C}_p$-analytic space X such that:

(i) If $X = \mathbb{G}_{\mathrm{m}}^{\mathrm{an}}$, then

$$^{\mathrm{BC}}\!\!\int_1^x \frac{\mathrm{d}T}{T} = \mathrm{Log}(x)$$

for a previously fixed branch of the logarithm.

(ii) If $f : X \longrightarrow Y$ is a morphism of smooth $\mathbb{C}_p$-analytic spaces and $\omega \in Z^1_{\mathrm{dR}}(Y)$, then

$$^{\mathrm{BC}}\!\!\int_\gamma f^*\omega = {}^{\mathrm{BC}}\!\!\int_{f(\gamma)} \omega.$$

(iii) Condition (i) from Definition 3.2.1 holds for any open subdomain $U \subseteq X$, which means: If $U \subseteq X$ is an open subdomain, and $\omega = \mathrm{d}f$ with f analytic on U, then

$$\int_\gamma \omega = f(\gamma(1)) - f(\gamma(0))$$

for all $\gamma : [0,1] \longrightarrow U$.

Remark 5.2.2. In section 5.1.3 it was explained that Coleman constructed a ring, particularly a filtered $\mathcal{O}(U_\zeta)$-algebra

$$A^\infty(U_\zeta) \subseteq \mathfrak{N}(U_\zeta),$$

whose de Rham complex

$$0 \longrightarrow A^\infty(U_\zeta) \xrightarrow{\mathrm{d}} A^\infty(U_\zeta)\,\mathrm{d}z \xrightarrow{\mathrm{d}} ...$$

is exact at $A^\infty(U_\zeta)\,\mathrm{d}z$ (see for Corollary 5.1.42). Berkovich took a similar strategy but was able to construct a filtered $\mathcal{O}_X$-algebra $S_X \subseteq \mathfrak{N}_X$ that is defined for a smooth K-analytic space X, where K is a closed subfield of $\mathbb{C}_p$. Let K be a closed subfield of $\mathbb{C}_p$ for the rest of this section.
Since it is not possible to elaborate the details of the book *Integration of One-forms on P-adic analytic spaces*, [Ber07], at least the most important constructions of Berkovich will be sketched and compared to the older approach of Coleman.

Comparison 1: **Space**

Coleman took the $\mathbb{C}_p$-analytic curve X and covered it by basic wide open subdomains that were, in contrast to the definition in section 5.1, defined in the language of rigid analysis and not in Berkovich language as was done in this paper. Then he defined the integration on the basic wide open subdomain that can be extended to X (see for Definition 5.1.51).
Berkovich defined his integration theory more generally on a smooth K-analytic space. Under certain conditions, described in the beginning of Chapter 4, [Ber07], he was able to connect two K-rational points of the K-analytic space by smooth K-analytic curves. This allowed him to reduce some problems to the one-dimensional case.

Comparison 2: **Definition of locally analytic functions**

Coleman considered locally analytic functions that are locally given as a power series and are only defined for $\mathbb{C}_p$-rational points.
Berkovich improved the definition of this class of functions and defined naive analytic functions (Definition 5.1.27). He noted that the original analogue, in the language of rigid analysis, of all the functions of $A^\infty(U_\zeta)$ can be obtained by restriction of a naive analytic function on U_ζ to $\mathbb{C}_p$-rational points (top of p. 3, [Ber07]).

Comparison 3: **Branch of the logarithm**

Coleman defined the branch of the logarithm, as explained in section 5.1.1, on $\mathbb{C}_p^\times$. Before defining the filtered $\mathcal{O}(U_\zeta)$-algebra $A^\infty(U_\zeta) \subseteq \mathfrak{N}_{U_\zeta}$, he fixed a certain, arbitrarily chosen, branch of the logarithm.
A main difference in the Berkovich-Coleman integral to the original approach of Coleman is that Berkovich, for the construction of the integral functions, does not choose a branch of the logarithm at the beginning but considers all branches at the same time by creating

a variable $\lambda := \mathrm{Log}(p)$. Later, in Theorem 9.1.1, [Ber07] and just before defining the integral, he fixed one branch of the logarithm.

Comparison 4: **Construction of the integral**

Coleman accomplished fixing the value of an integral, whose path from P to Q passes one vertex, by constructing the logarithmic F-crystal $A^{\infty}(U_{\zeta})$, which fulfils the uniqueness principle. This is very essential for fixing the value of the integral

$$\int_P^Q \omega = F_2(Q) - F_1(P)$$

where F_1 and F_2 are primitives on the different open annuli or open balls. Afterwards, Coleman extended the integral to curves by covering any curve with basic wide open subdomains.
Berkovich showed in Chapter 7, [Ber07] the existence of a sheaf S_X^{λ} for a smooth K-analytic space X and a branch of the logarithm λ, which he developed to the filtered $\mathcal{O}_X$-algebra $S_X \subseteq \mathfrak{N}_X$ mentioned before. This sheaf provides much more antiderivatives than the analogue $A^{\infty}(U_{\zeta})$ of Coleman, with the result that Berkovich could define his integration theory for any closed differential one-form on X. He proved the existence and uniqueness of the Berkovich-Coleman integral in Theorem 9.1.1, [Ber07]. The structure of its proof is very similar to Coleman's one. A small difference is that in Berkovich's proof functoriality is used for finding a primitive of ω. The path is a continuous map

$$\gamma : [0,1] \longrightarrow X$$

to the $\mathbb{C}_p$-analytic space X. He pulled back the closed differential one-form ω along this path and used the contractibility of the interval $[0,1]$ to show the existence of a primitive g of $\gamma^*\omega$. Then he defined

$${}^{\mathrm{BC}}\!\!\int_{\gamma} \omega := g(1) - g(0).$$

Functoriality delivers property (iii) of Definition 5.2.1, which is a generalization of property (i) of Definition 3.2.1. If it holds $\omega = df$, one can choose g such that it fulfils $g = \gamma^* f$ and it can be calculated

$${}^{\mathrm{BC}}\!\!\int_{\gamma} \omega = g(1) - g(0) = (\gamma^* f)(1) - (\gamma^* f)(0) = f(\gamma(1)) - f(\gamma(0)).$$

Theorem 5.2.3. *The Berkovich-Coleman integration theory exists and is unique.*

Proof. Theorem 9.1.1, [Ber07]. □

6 Comparing the integrals

6.1 The tropical Abel-Jacobi map

Let K be a field that is algebraically closed and complete with respect to a nontrivial, non-archimedean valuation $\mathrm{val} : K \longrightarrow \mathbb{R} \cup \{\infty\}$ and X be a smooth, proper, connected K-curve, along with a semistable model $\mathcal{X}$. Since the skeleton was just defined for $K = \mathbb{C}_p$, this will be assumed in the following.

Construction 6.1.1. Restrict $[.,.]$ to $M := H_1(\Gamma, \mathbb{Z}) \subseteq C_1(\Gamma, \mathbb{Z})$. With

$$N_{\mathbb{R}} := \mathrm{Hom}(H_1(\Gamma, \mathbb{Z}), \mathbb{R}) = H^1(\Gamma, \mathbb{R})$$

one gets a homomorphism

$$\begin{aligned} \eta' : M &\longrightarrow N_{\mathbb{R}}, \\ m &\longmapsto [., m]. \end{aligned}$$

This is a homomorphism, as $[.,.]$ is symmetric and bilinear. As $M = H_1(\Gamma, \mathbb{Z})$ is a free abelian group, $\eta'(M)$ is a lattice in $N_{\mathbb{R}}$. If the rank of M is $g \in \mathbb{N}\backslash\{0\}$, it holds $N_{\mathbb{R}} \cong \mathbb{R}^g$.

Definition 6.1.2. The **Jacobian** of the connected, antisymmetric, weighted graph Γ is defined as

$$\mathrm{Jac}(\Gamma) := N_{\mathbb{R}}/\eta'(M).$$

The Jacobian of a skeleton $\Gamma_{\mathcal{X}}$ is defined to be the Jacobian of its corresponding graph Γ and denoted by $\mathrm{Jac}(\Gamma_{\mathcal{X}})$.

Construction 6.1.3. Let $x_0 \in \Gamma_{\mathcal{X}}$ be a point of the skeleton that is not a vertex. Then x_0 lies in an open annulus A whose skeleton can be identified with $(0, -\log(\rho))$, where $\rho \in |\mathbb{C}_p^{\times}|$. Hence x_0 corresponds uniquely to an element

$$r_0 \in (0, -\log(\rho)).$$

The open annulus lies between two vertices $x_1, x_2 \in \Gamma_{\mathcal{X}}$, which can be identified with the ends of the closed interval $[0, -\log(\rho)]$. Let $x_1 \in \Gamma_{\mathcal{X}}$ be the vertex corresponding to 0 and $x_2 \in \Gamma_{\mathcal{X}}$ be the vertex corresponding to $-\log(\rho)$.
In the graph Γ, the vertices $x_1, x_2 \in \Gamma_{\mathcal{X}}$ correspond to vertices $V_1, V_2 \in \Gamma$. The aim is to create a new graph Γ_{x_0} by taking Γ and doing the following. Add a vertex V_0 to Γ_{x_0} and replace the edge e, corresponding to the skeleton of the open annulus A with length $l(e) = -\log(\rho)$, by two edges $e_1, e_2 \in E(\Gamma_{x_0})$. If $\iota_{\mathrm{ass}}(e) = (V_1, V_2)$ the edges will be defined by

$\iota_{\text{ass}}(e_1) = (V_1, V_0)$, $\iota_{\text{ass}}(e_2) = (V_0, V_2)$, $l(e_1) = r_0 > 0$ and $l(e_2) = -\log(\rho) - r_0 > 0$. If $\iota_{\text{ass}}(e) = (V_2, V_1)$, tail and head vertices will be changed. This delivers a new connected, antisymmetric, weighted graph Γ_{x_0}. The notation $\Gamma_{x_0,x_0'} = (\Gamma_{x_0})_{x_0'}$ will be used in the following.

Lemma 6.1.4. *Γ_{x_0} from Construction 6.1.3 refines Γ.*

Proof. There exists an injection

$$a : V(\Gamma) \hookrightarrow V(\Gamma_{x_0})$$

and a surjection

$$b : E(\Gamma_{x_0}) \twoheadrightarrow E(\Gamma)$$

with $b(e_1) = b(e_2) = e$ where the edges from Construction 6.1.3 are meant. For $e \in \Gamma$, there exist the vertices $V_0, V_1, V_2 \in V(\Gamma_{x_0})$ and the edges $e_1, e_2 \in E(\Gamma_{x_0})$, such that

(i) $b^{-1}(e) = \{e_1, e_2\}$,
(ii) $l(e_1) + l(e_2) = l(e)$ and
(iii) $\iota_{\text{ass}}(e_1) = (V_1, V_0)$, $\iota_{\text{ass}}(e_2) = (V_0, V_2)$

where ι_{ass} is the edge assignment map with respect to Γ_{x_0}. Therefore Γ_{x_0} refines Γ. $\square$

Definition 6.1.5. The **tropical Abel-Jacobi map** with respect to a point $x_0 \in \Gamma_{\mathcal{X}}$ is the function

$$\begin{aligned} \iota' : \Gamma_{\mathcal{X}} &\longrightarrow \operatorname{Jac}(\Gamma_{\mathcal{X}}), \\ x &\longmapsto \overline{[.,p]} \end{aligned}$$

where $p \in C_1(\Gamma_{x_0,x}, \mathbb{Z})$ is a path in $\Gamma_{x_0,x}$ from the vertex, corresponding to x_0, to the vertex, corresponding to x.

Remark 6.1.6. A priori $[.,p]$ is an element of $\operatorname{Hom}(H_1(\Gamma_{x_0,x}, \mathbb{Z}), \mathbb{R}) = H^1(\Gamma_{x_0,x}, \mathbb{R})$ and not of $N_{\mathbb{R}} = \operatorname{Hom}(H_1(\Gamma, \mathbb{Z}), \mathbb{R}) = H^1(\Gamma, \mathbb{R})$. Restricting the homomorphism $[.,p]$ to $H_1(\Gamma, \mathbb{Z})$ by

$$\begin{aligned} [.,p] : H_1(\Gamma, \mathbb{Z}) &\longrightarrow \mathbb{R}, \\ m &\longmapsto [m_{\text{refine}}, p] \end{aligned}$$

allows to consider $[.,p]$ as an element of $N_{\mathbb{R}} = \operatorname{Hom}(H_1(\Gamma, \mathbb{Z}), \mathbb{R}) = H^1(\Gamma, \mathbb{R})$.

Lemma 6.1.7. *The tropical Abel-Jacobi map from Definition 6.1.5 is well-defined.*

Proof. Let $p_1, p_2 \in C_1(\Gamma_{x_0,x}, \mathbb{Z})$ be two paths as required in Definition 6.1.5. Then $p_1 - p_2 \in C_1(\Gamma_{x_0,x}, \mathbb{Z})$ and

$$\mathrm{d}^*(p_1 - p_2)\,(V) = 0$$

for all $V \in V(\Gamma_{x_0,x})$. Hence $p_1 - p_2 \in H_1(\Gamma_{x_0,x}, \mathbb{Z})$ and $[., p_1 - p_2] \in \eta'(H_1(\Gamma_{x_0,x}, \mathbb{Z}))$. Finally

$$\overline{[., p_1]} = \overline{[., p_1 - p_2 + p_2]} = \overline{[., p_1 - p_2]} + \overline{[., p_2]} = \overline{0} + \overline{[., p_2]} = \overline{[., p_2]}$$

and ι' is well-defined. □

Example 6.1.8. Consider a connected, antisymmetric, weighted graph Γ associated to the analytification of the curve X.

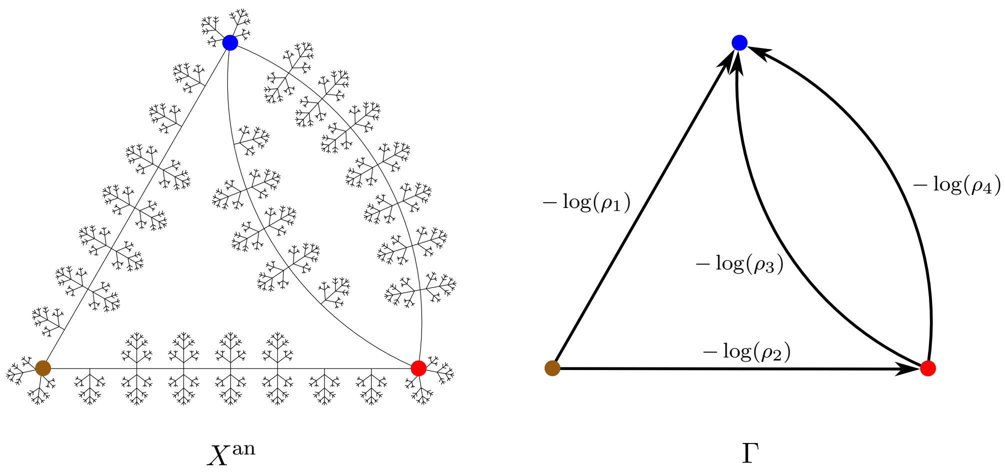


Figure 6.1: Γ associated to $\Gamma_{\mathcal{X}}$ with $\mathcal{X}_\eta \cong X^{\mathrm{an}}$

Define $a := -\log(\rho_1)$, $b := -\log(\rho_2)$, $c := -\log(\rho_3)$ and $d := -\log(\rho_4)$. Furthermore, the corresponding edges are named e_a, e_b, e_c and e_d.
$M = H_1(\Gamma, \mathbb{Z})$ are the integral 1-cycles of the graph Γ. Each two of the three integral 1-cycles

$$\begin{aligned} m_1 &= e_c - e_a + e_b \\ m_2 &= e_c - e_d \\ m_3 &= e_d - e_a + e_b \end{aligned}$$

form a basis of the lattice M (see Figure 6.2). m_1, m_2 and m_3 generate M, and because of $m_3 = m_1 - m_2$, the set $\{m_1, m_2\}$ is a basis of the rank 2 lattice M. Hence $[., m_1]$ and $[., m_2]$ form a basis of $N_{\mathbb{R}}$ as the edge length pairing is non-degenerate. Fixing the basis $\{m_1, m_2\}$ of M yields the canonical isomorphism

$$\begin{aligned} \alpha : N_{\mathbb{R}} &\longrightarrow \mathbb{R}^2, \\ \varphi &\longmapsto (\varphi(m_1), \varphi(m_2)). \end{aligned}$$

The property

$$\eta'(M) = \eta'(m_1\mathbb{Z} + m_2\mathbb{Z}) = \eta'(m_1)\mathbb{Z} + \eta'(m_2)\mathbb{Z}$$

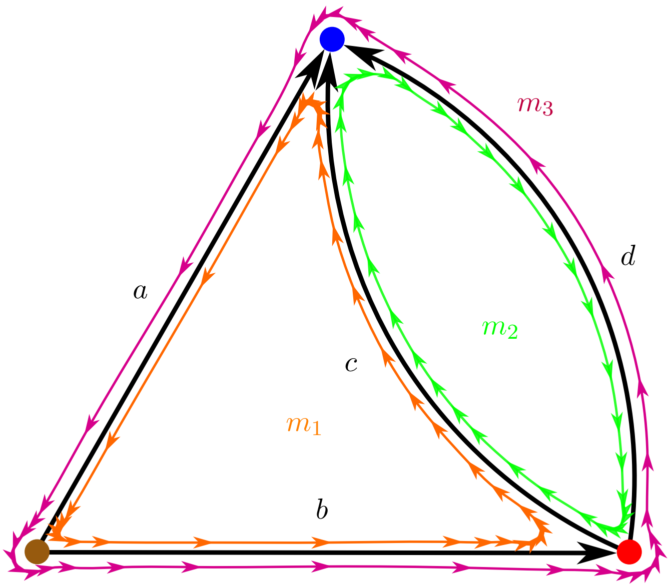


Figure 6.2: Three integral 1-cycles of the graph Γ

gives

$$\begin{aligned}\mathrm{Jac}(\Gamma_{\mathcal{X}}) = \mathrm{Jac}(\Gamma) &= N_{\mathbb{R}}/\eta'(M)\\ &\cong \alpha(N_{\mathbb{R}})/\alpha(\eta'(M))\\ &= \mathbb{R}^2/\alpha\left(\eta'(m_1)\mathbb{Z} + \eta'(m_2)\mathbb{Z}\right)\\ &= \mathbb{R}^2/\left\{\alpha(\eta'(m_1))\mathbb{Z} + \alpha(\eta'(m_2))\mathbb{Z}\right\}.\end{aligned}$$

With

$$\begin{aligned}\alpha(\eta'(m_1)) &= \alpha([.,m_1])\\ &= ([m_1,m_1],[m_2,m_1])\\ &= (a+b+c,c)\end{aligned}$$

and

$$\begin{aligned}\alpha(\eta'(m_2)) &= \alpha([.,m_2])\\ &= ([m_1,m_2],[m_2,m_2])\\ &= (c,c+d)\end{aligned}$$

follows

$$\mathrm{Jac}(\Gamma_{\mathcal{X}}) \cong \mathbb{R}^2/\left\{(a+b+c,c)\,\mathbb{Z} + (c,c+d)\,\mathbb{Z}\right\}.$$

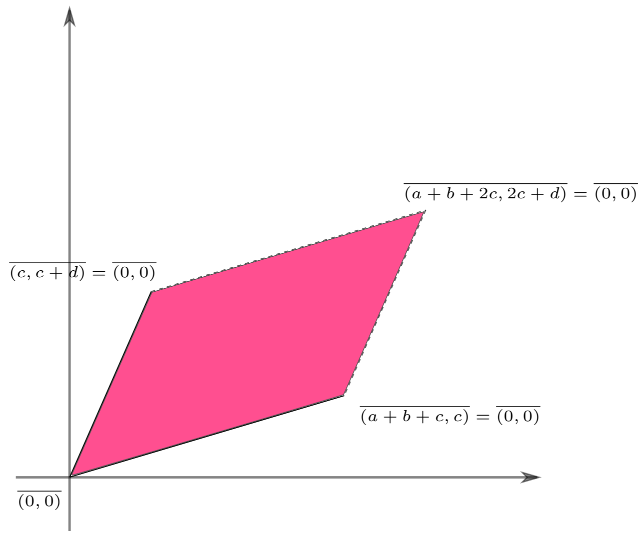


Figure 6.3: The purple marked area is $\mathrm{Jac}(\Gamma_{\mathcal{X}})$

The next goal is to calculate the tropical Abel-Jacobi map for a point $x_1 \in \Gamma_{\mathcal{X}}$. Choose the base point to be the red vertex and denote it by $x_0 \in \Gamma$.
Since $x_1 \in \Gamma_{\mathcal{X}}$ in general does not correspond to a vertex in Γ, one has to refine Γ by x_1 and gets Γ_{x_1}. The new vertex, corresponding to x_1, is marked in yellow.

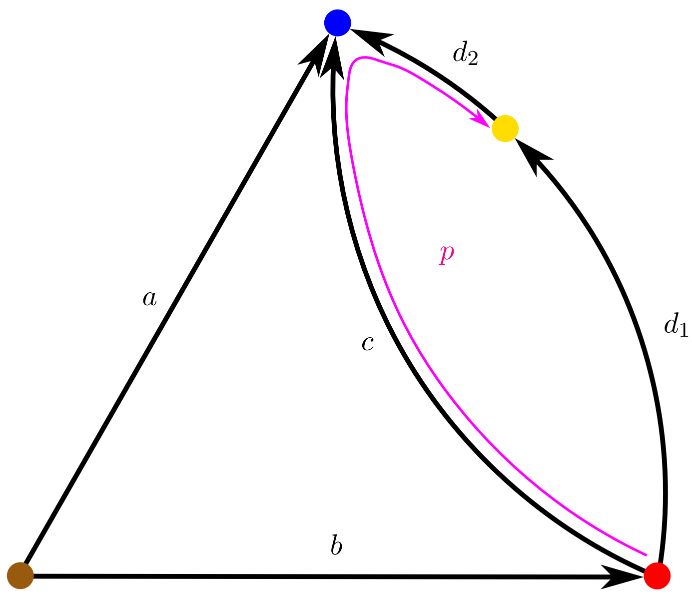


Figure 6.4: Refinement of Γ

Choose a path p from x_0 to x_1, that means from the red to the yellow vertex. One possibility is $p = e_c - e_{d_2}$. Then one has

$$\begin{aligned} \iota' : \Gamma_{\mathcal{X}} &\longrightarrow \mathrm{Jac}(\Gamma_{\mathcal{X}}), \\ x_1 &\longmapsto \overline{\alpha^{-1}\left(([p, m_1], [p, m_2])\right)}. \end{aligned}$$

This map yields for x_1:

$$\begin{aligned} \iota'(x_1) &= \overline{\alpha^{-1}\left(([p, m_1], [p, m_2])\right)} \\ &= \overline{\alpha^{-1}\left((c, c + d_2)\right)}. \end{aligned}$$

Under the identification $N_{\mathbb{R}} \cong \mathbb{R}^2$, this becomes

$$\iota'(x_1) = \overline{(c, c + d_2)}.$$

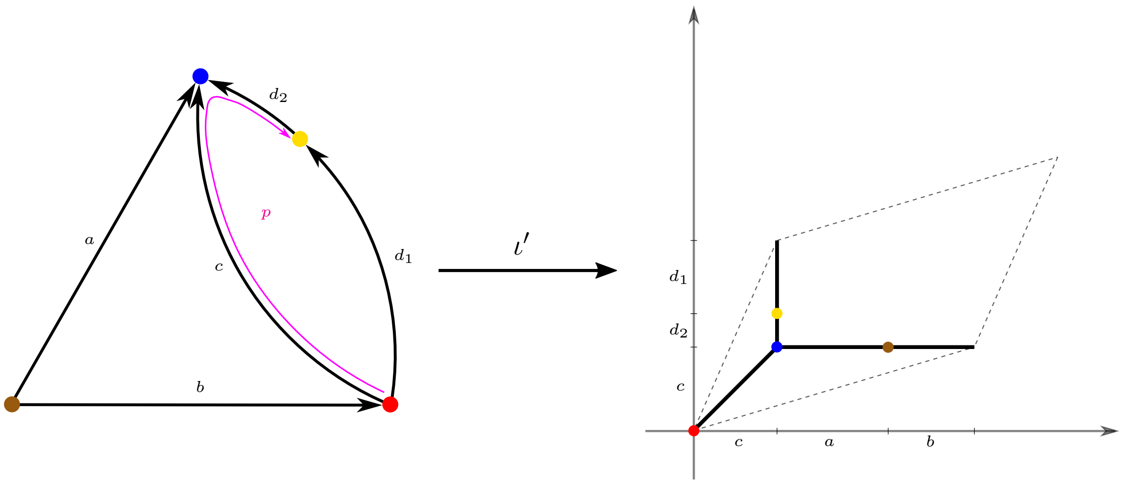


Figure 6.5: The tropical Abel-Jacobi map ι'

The definition of the Abel-Jacobi map is independent of the choice of the path p, as this choice can only differ by a $\mathbb{Z}$-linear combination of m_1 and m_2, but they map to zero in $\mathrm{Jac}(\Gamma_{\mathcal{X}})$.
If one chooses, for instance, the path $p' = d_1$, it holds $p' = p - m_2$, which gives

$$\begin{aligned} \iota'(x_1) &= \overline{\alpha^{-1}\left(([p', m_1], [p', m_2])\right)} \\ &= \overline{\alpha^{-1}\left(([p, m_1] - [m_2, m_1], [p, m_2] - [m_2, m_2])\right)} \\ &= \overline{\alpha^{-1}\left(([p, m_1], [p, m_2])\right)} - \overline{\alpha^{-1}\left(([m_2, m_1], [m_2, m_2])\right)} \\ &= \overline{\alpha^{-1}\left(([p, m_1], [p, m_2])\right)} - \overline{[m_2, .]} \\ &= \overline{\alpha^{-1}\left(([p, m_1], [p, m_2])\right)} - \overline{[., m_2]} \\ &= \overline{\alpha^{-1}\left(([p, m_1], [p, m_2])\right)} - \overline{\eta'(m_2)} \\ &= \overline{\alpha^{-1}\left(([p, m_1], [p, m_2])\right)} - \overline{0} \\ &= \overline{\alpha^{-1}\left(([p, m_1], [p, m_2])\right)}, \end{aligned}$$

which is the same result as before.
Eventually one can lift the tropical Abel-Jacobi map ι' to $\tilde{\iota}'$. Therefore one has to find covers of Γ and $\mathrm{Jac}(\Gamma)$. The cover of the latter is $N_{\mathbb{R}}$ but the cover of Γ may be more difficult. In this case one gets the following map, which will be important later on:

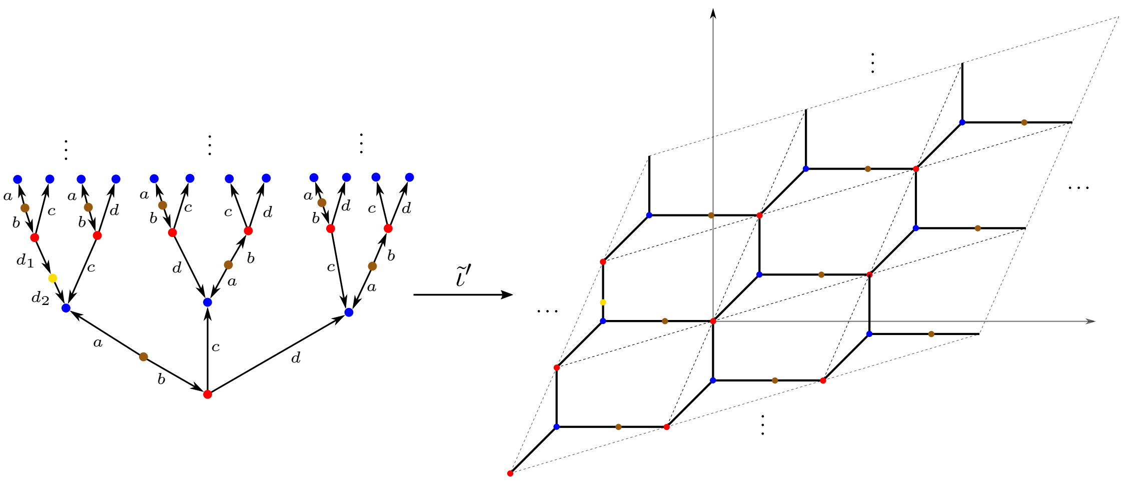


Figure 6.6: The cover $\tilde{\iota}'$ of the tropical Abel-Jacobi map

Remark 6.1.9. In the literature you can find two different definitions of the Jacobian of a connected, antisymmetric, weighted graph Γ. The first one has been introduced in Definition 6.1.2. In the following, the second one is going to be described, and afterwards it will be shown that both are canonically isomorphic.

Definition 6.1.10. The **divisor group** $\mathrm{Div}_{\mathbb{R}}(\Gamma)$ of Γ is defined to be $C_0(\Gamma, \mathbb{R})$. The degree of a divisor $D \in \mathrm{Div}_{\mathbb{R}}(\Gamma)$ is defined to be

$$\deg(D) = \sum_{V \in V(\Gamma)} D(V).$$

Furthermore it is defined

$$\begin{aligned} \mathrm{Div}^0_{\mathbb{R}}(\Gamma) &:= \{D \in \mathrm{Div}_{\mathbb{R}}(\Gamma) : \deg(D) = 0\} \\ \mathrm{Prin}_{\mathbb{R}}(\Gamma) &:= \mathrm{d}^*(\mathrm{Im}(\mathrm{d}))\,. \end{aligned}$$

Finally

$$\mathrm{Pic}^0_{\mathbb{R}}(\Gamma) := \mathrm{Div}^0_{\mathbb{R}}(\Gamma) / \mathrm{Prin}_{\mathbb{R}}(\Gamma).$$

Lemma 6.1.11. $\mathrm{Pic}^0_{\mathbb{R}}(\Gamma)$ *is a well-defined commutative group.*

Proof. First show $\mathrm{Prin}_{\mathbb{R}}(\Gamma) \subseteq \mathrm{Div}^0_{\mathbb{R}}(\Gamma)$. Let

$$f = \sum_{V \in V(\Gamma)} n_V V \in \mathrm{Prin}_{\mathbb{R}}(\Gamma) \subseteq C_0(\Gamma, \mathbb{R}).$$

Then there exists an $\alpha = \sum_{e \in E(\Gamma)} n_e e \in \mathrm{Im}(\mathrm{d})$ with $\mathrm{d}^*\alpha = f$. Consequently there exists an

$$g = \sum_{V \in V(\Gamma)} m_V V \in C_0(\Gamma, \mathbb{R})$$

such that $\mathrm{d}g = \alpha$. Therefore

$$\begin{aligned} \mathrm{d}g : E(\Gamma) &\longrightarrow \mathbb{R}, \\ e &\longmapsto \frac{m_{e^+} - m_{e^-}}{l(e)}. \end{aligned}$$

This leads to

$$\begin{aligned} f = \mathrm{d}^*\mathrm{d}g : V(\Gamma) &\longrightarrow \mathbb{R}, \\ V &\longmapsto \sum_{\substack{e \in E(\Gamma) \\ e^+ = V}} \mathrm{d}g(e) - \sum_{\substack{e \in E(\Gamma) \\ e^- = V}} \mathrm{d}g(e) \\ &= \sum_{\substack{e \in E(\Gamma) \\ e^+ = V}} \frac{m_{e^+} - m_{e^-}}{l(e)} - \sum_{\substack{e \in E(\Gamma) \\ e^- = V}} \frac{m_{e^+} - m_{e^-}}{l(e)} \end{aligned}$$

and

$$\begin{aligned} \deg(f) &= \sum_{V \in V(\Gamma)} \left(\sum_{\substack{e \in E(\Gamma) \\ e^+ = V}} \frac{m_{e^+} - m_{e^-}}{l(e)} - \sum_{\substack{e \in E(\Gamma) \\ e^- = V}} \frac{m_{e^+} - m_{e^-}}{l(e)} \right) \\ &= \sum_{e \in E(\Gamma)} \frac{m_{e^+} - m_{e^-}}{l(e)} - \sum_{e \in E(\Gamma)} \frac{m_{e^+} - m_{e^-}}{l(e)} \\ &= 0. \end{aligned}$$

This shows $f \in \mathrm{Div}^0_{\mathbb{R}}(\Gamma)$.
Since d and d^* are linear, it is clear that $\mathrm{Prin}_{\mathbb{R}}(\Gamma)$ and $\mathrm{Div}^0_{\mathbb{R}}(\Gamma)$ are subgroups of $\mathrm{Div}_{\mathbb{R}}(\Gamma)$. Moreover $\mathrm{Div}_{\mathbb{R}}(\Gamma)$ is commutative, hence any subgroup is normal, which means

$$\mathrm{Pic}^0_{\mathbb{R}}(\Gamma) = \mathrm{Div}^0_{\mathbb{R}}(\Gamma) / \mathrm{Prin}_{\mathbb{R}}(\Gamma)$$

is a commutative group. ∎

Lemma 6.1.12. *The map*

$$\begin{aligned}\eta'_{\mathbb{R}} : H_1(\Gamma, \mathbb{R}) &\longrightarrow N_{\mathbb{R}} = \mathrm{Hom}(H_1(\Gamma, \mathbb{Z}), \mathbb{R})\\ \alpha &\longmapsto [., \alpha]\end{aligned}$$

is a group isomorphism. Note that the restriction of $\eta'_{\mathbb{R}}$ to $H_1(\Gamma, \mathbb{R})$ is η'.

Proof. Group homomorphism: Let $\alpha_1 = \sum_{e\in E(\Gamma)} n_e e,\ \alpha_2 = \sum_{e\in E(\Gamma)} m_e e \in H_1(\Gamma, \mathbb{R})$. Then it holds

$$\begin{aligned}\eta'_{\mathbb{R}}(\alpha_1 + \alpha_2) &= \eta'_{\mathbb{R}}\left(\sum_{e\in E(\Gamma)} (n_e + m_e)e\right)\\ &= \left[\,.\,, \sum_{e\in E(\Gamma)} (n_e + m_e)e\right]\\ &= \sum_{e\in E(\Gamma)} \ .\ \cdot (n_e + m_e)l(e)\\ &= \sum_{e\in E(\Gamma)} \ .\ \cdot n_e l(e) + \sum_{e\in E(\Gamma)} \ .\ \cdot m_e l(e)\\ &= \left[\,.\,, \sum_{e\in E(\Gamma)} n_e e\right] + \left[\,.\,, \sum_{e\in E(\Gamma)} m_e e\right]\\ &= \eta'_{\mathbb{R}}(\alpha_1) + \eta'_{\mathbb{R}}(\alpha_2).\end{aligned}$$

Injectivity: Take an $\alpha = \sum_{e\in E(\Gamma)} n_e e \in \ker(\sigma)$, ergo $\eta'_{\mathbb{R}}(\alpha) = [., \alpha] = 0$. This means that, for all $\beta = \sum_{e\in E(\Gamma)} m_e e \in H_1(\Gamma, \mathbb{R})$, it follows

$$[\beta, \alpha] = \sum_{e\in E(\Gamma)} m_e n_e l(e) = 0,$$

and particularly for any edge $\beta = \alpha$ one has

$$[\alpha, \alpha] = \sum_{e\in E(\Gamma)} (n_e)^2 l(e) = 0,$$

ergo $n_e = 0$ for all edges $e \in E(\Gamma)$, which means $\alpha = 0$.

Surjectivity: Let μ be a homomorphism from $H_1(\Gamma, \mathbb{Z})$ to $\mathbb{R}$, meaning that

$$\mu \in N_{\mathbb{R}} = \mathrm{Hom}(H_1(\Gamma, \mathbb{Z}), \mathbb{R}).$$

Since $C_1(\Gamma, \mathbb{Z})$ is a free $\mathbb{Z}$-module and $H_1(\Gamma, \mathbb{Z}) = \ker(\mathrm{d}^*)$ is a submodule, it must be free, too. This means that there exists a basis $\{m_i : i \in I\}$ of $H_1(\Gamma, \mathbb{Z})$, where I denotes the index set. Consequently the subset

$$\{[., m_i] : i \in I\} \subseteq N_{\mathbb{R}} = \mathrm{Hom}(H_1(\Gamma, \mathbb{Z}), \mathbb{R})$$

is a basis of $\mathrm{Hom}(H_1(\Gamma,\mathbb{Z}),\mathbb{R})$, which is a real vector space. Hence μ can be written as a linear combination of the $[.,m_i]$, namely $\sum_{i\in J} r_i[.,m_i]$ where J is a finite subset of I. The equality

$$\mu = \sum_{i\in J} r_i[.,m_i] = \left[\ .\ ,\sum_{i\in J} r_i m_i\right]$$

gives

$$\eta'_{\mathbb{R}}\left(\sum_{i\in J} r_i m_i\right) = \mu$$

with $\sum_{i\in J} r_i m_i \in H_1(\Gamma,\mathbb{R})$. □

Corollary 6.1.13. $\mathrm{Jac}(\Gamma) \cong H_1(\Gamma,\mathbb{R})/H_1(\Gamma,\mathbb{Z})$*, and if the basis of the real vector space* $H_1(\Gamma,\mathbb{R})$ *has cardinality* $g \in \mathbb{N}\backslash\{0\}$*, it follows*

$$\mathrm{Jac}(\Gamma) \cong (\mathbb{R}/\mathbb{Z})^g.$$

Proof. It holds

$$\begin{aligned}\eta'_{\mathbb{R}}\left(H_1(\Gamma,\mathbb{R})/H_1(\Gamma,\mathbb{Z})\right) &= \eta'_{\mathbb{R}}\left(H_1(\Gamma,\mathbb{R})\right)/\eta'_{\mathbb{R}}\left(H_1(\Gamma,\mathbb{Z})\right)\\ &= N_{\mathbb{R}}/\eta'\left(H_1(\Gamma,\mathbb{Z})\right)\\ &= \mathrm{Jac}(\Gamma)\end{aligned}$$

and, after choosing a $\mathbb{Z}$-basis $\{m_1,...,m_g\}$ of the $\mathbb{Z}$-module $H_1(\Gamma,\mathbb{Z})$,

$$\begin{aligned}H_1(\Gamma,\mathbb{R})/H_1(\Gamma,\mathbb{Z}) &= \left(H_1(\Gamma,\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{R}\right)/\left(H_1(\Gamma,\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Z}\right)\\ &= \left((\mathbb{Z}m_1+...+\mathbb{Z}m_g)\otimes_{\mathbb{Z}}\mathbb{R}\right)/\left((\mathbb{Z}m_1+...+\mathbb{Z}m_g)\otimes_{\mathbb{Z}}\mathbb{Z}\right)\\ &\cong (\mathbb{R}/\mathbb{Z})^g.\end{aligned}$$

□

Theorem 6.1.14. *There exists a canonical isomorphism between* $\mathrm{Jac}(\Gamma)$ *and* $\mathrm{Pic}^0_{\mathbb{R}}(\Gamma)$*.*

Proof. Baker and Faber proved this in Theorem 3.4, [BF10], but they used a different notation. Firstly, it is proved for a tropical curve. By the preface of section 3, [BF10], one can uniquely associate a connected, antisymmetric, weighted graph with a tropical curve.

Secondly, Baker and Faber defined the Jacobian of Γ as $\Omega(\Gamma)^*/H_1(\Gamma,\mathbb{Z})$, where $\Omega(\Gamma) \subseteq C_1(\Gamma,\mathbb{Z})$ is the “space of harmonic 1-forms” (Preface section 2, (2.2), [BF10]). Comparing with the notion of this paper, that is exactly $\ker(\mathrm{d}^*)$, as harmonic for an element $\alpha \in C_1(\Gamma,\mathbb{Z})$ means just $\mathrm{d}^*\alpha = 0$. Hence it holds

$$\Omega(\Gamma)^* = \ker(\mathrm{d}^*)^* = H_1(\Gamma,\mathbb{Z})^* = \mathrm{Hom}(H_1(\Gamma,\mathbb{Z}),\mathbb{R}) = N_{\mathbb{R}}.$$

Since $\eta'_{\mathbb{R}}$ is an isomorphism, $H_1(\Gamma, \mathbb{Z})$ and $\eta'_{\mathbb{R}}(H_1(\Gamma, \mathbb{Z})) = \eta'(H_1(\Gamma, \mathbb{Z}))$ are isomorphic. It follows

$$\Omega(\Gamma)^*/H_1(\Gamma, \mathbb{Z}) \cong N_{\mathbb{R}}/\eta'(H_1(\Gamma, \mathbb{Z})) = \mathrm{Jac}(\Gamma).$$

Therefore the Jacobian defined by Baker and Faber, and the Jacobian defined in Definition 6.1.2 are isomorphic. This permits using Theorem 3.4, [BF10] for proving the above theorem. □

Remark 6.1.15. For defining $\mathrm{Pic}^0_{\mathbb{R}}(\Gamma)$ to be the Jacobian of Γ, there exists another definition of the tropical Abel-Jacobi map with respect to a point $x_0 \in \Gamma$:

$$\begin{aligned} \iota' : \Gamma &\longrightarrow \mathrm{Jac}(\Gamma), \\ x &\longmapsto \overline{[x] - [x_0]} \end{aligned}$$

One takes the divisor $[x]$ of $x \in \Gamma$. This is always possible if $\mathrm{Div}(\Gamma)$ is considered as the group of the divisors for any point $x \in \Gamma$, or just for the vertices as Baker and Faber in [BF10]. Both versions can be found in the literature. But by refining Γ, they coincide. Then $[x] - [x_0] \in \mathrm{Div}^0_{\mathbb{R}}(\Gamma)$ and

$$\overline{[x] - [x_0]} \in \mathrm{Div}^0_{\mathbb{R}}(\Gamma)/\,\mathrm{Prin}_{\mathbb{R}}(\Gamma)$$

denotes the corresponding equivalence class, which is an element of the Jacobian of Γ. Here it even holds $[x]-[x_0] \in \mathrm{Div}^0(\Gamma)$ as the coefficients are from $\mathbb{Z}$. Hence one can identify $\overline{[x] - [x_0]} \in \mathrm{Pic}^0_{\mathbb{R}}(\Gamma)$ explicitly with an element from $N_{\mathbb{R}}/\eta'(H_1(\Gamma, \mathbb{Z})) = \mathrm{Jac}(\Gamma)$, with respect to the canonical isomorphism of Theorem 6.1.14. From Theorem 2.8, [BF10] and the previous remark, directly after Definition 2.6, [BF10], one knows that the canonical isomorphism descends from $\mathrm{d}^* : C_1(\Gamma, \mathbb{Z}) \longrightarrow C_0(\Gamma, \mathbb{Z})$. Let $p \in C_1(\Gamma, \mathbb{Z})$ be a path from $[x_0]$ to $[x]$, both considered as vertices of Γ_{x,x_0}, then one has

$$\mathrm{d}^* p = [x] - [x_0].$$

By the proof of Lemma 2.7, [BF10], $\overline{[.,p]}$ is the corresponding element of $\mathrm{Jac}(\Gamma) = N_{\mathbb{R}}/\eta'(H_1(\Gamma, \mathbb{Z}))$. This is exactly how the tropical Abel-Jacobi map is defined in Definition 6.1.5.
As a consequence, both definitions are equivalent. In the following, the version of Definition 6.1.5 is preferred.

Theorem 6.1.16. *Consider the lift $\tilde{\iota}' : \tilde{\Gamma} \longrightarrow N_{\mathbb{R}}$ of ι' with respect to the universal cover $\pi : \tilde{\Gamma} \longrightarrow \Gamma$. Let $\tilde{e} \subseteq \tilde{\Gamma}$ be an edge, and let $e \subseteq \Gamma$ be its image. Then it holds:*

(i) If $\Gamma \backslash e$ is disconnected, then $\tilde{\iota}'$ is constant on $\tilde{e}$.
(ii) If $\Gamma \backslash e$ is connected, then $\tilde{\iota}'$ is affine-linear on $\tilde{e}$ with rational slope.

Proof. Firstly, the lift $\tilde{\iota}' : \tilde{\Gamma} \longrightarrow N_{\mathbb{R}}$ of ι' to universal covers is given by $x \longmapsto [.,p]$, where $\tilde{p} \in C_1(\tilde{\Gamma}_{x_0,x}, \mathbb{Z})$ is a path in $\tilde{\Gamma}_{x_0,x}$ from the vertex corresponding to x_0, the chosen base point, to the vertex corresponding to x. Then p is defined as $\pi'(\tilde{p})$, where

$$\pi' : C_1(\tilde{\Gamma}, \mathbb{Z}) \longrightarrow C_1(\Gamma, \mathbb{Z}),$$
$$\tilde{\alpha} \longmapsto \sum_{e \in E(\Gamma)} \left(\sum_{\substack{\tilde{e} \in E(\tilde{\Gamma}) \\ \pi(\tilde{e}) = e}} \tilde{\alpha}(\tilde{e}) \right) e$$

denotes the reduction of 1-chains with respect to π. It is a group homomorphism because

$$\begin{aligned}
\pi'(\tilde{\alpha} + \tilde{\beta}) &= \sum_{e \in E(\Gamma)} \left(\sum_{\substack{\tilde{e} \in E(\tilde{\Gamma}) \\ \pi(\tilde{e}) = e}} (\tilde{\alpha} + \tilde{\beta})(\tilde{e}) \right) e \\
&= \sum_{e \in E(\Gamma)} \left(\sum_{\substack{\tilde{e} \in E(\tilde{\Gamma}) \\ \pi(\tilde{e}) = e}} \tilde{\alpha}(\tilde{e}) + \sum_{\substack{\tilde{e} \in E(\tilde{\Gamma}) \\ \pi(\tilde{e}) = e}} \tilde{\beta}(\tilde{e}) \right) e \\
&= \sum_{e \in E(\Gamma)} \left(\sum_{\substack{\tilde{e} \in E(\tilde{\Gamma}) \\ \pi(\tilde{e}) = e}} \tilde{\alpha}(\tilde{e}) \right) e + \sum_{e \in E(\Gamma)} \left(\sum_{\substack{\tilde{e} \in E(\tilde{\Gamma}) \\ \pi(\tilde{e}) = e}} \tilde{\beta}(\tilde{e}) \right) e \\
&= \pi'(\tilde{\alpha}) + \pi'(\tilde{\beta})
\end{aligned}$$

for $\tilde{\alpha}, \tilde{\beta} \in C_1(\tilde{\Gamma}, \mathbb{Z})$.

(i) If $\Gamma \backslash e$ is disconnected, it consists of two connected subgraphs Γ_1 and Γ_2. The only connection between them is the edge e. Therefore an arbitrary cycle $m \in H_1(\Gamma, \mathbb{Z})$ has to pass e in positive direction just as often as in negative direction. Hence it holds $[e, m] = 0$. Let x_1, x_2 be points on $\tilde{e}$. It has to be shown

$$\tilde{\iota}'(x_1) = \tilde{\iota}'(x_2).$$

First refine $\tilde{\Gamma}$ such that there exist vertices $V_1, V_2 \in V(\tilde{\Gamma})$ corresponding to the points x_1, x_2. Denote the resulting edge between V_1 and V_2, which has the same orientation as $\tilde{e}$, by $\tilde{e}_{1,2}$. Then one has still

$$[e_{1,2}, m] = 0$$

for all $m \in H_1(\Gamma, \mathbb{Z})$, where $e_{1,2} \in E(\Gamma_{x_1,x_2})$ is the reduced edge of $\tilde{e}_{1,2}$. Notice that the path from V_1 to V_2 is $\tilde{e}_{1,2}$ or $-\tilde{e}_{1,2}$. Thus it will be written $\pm\tilde{e}_{1,2}$ for this path
Let $x_0 \in \tilde{\Gamma}$ be an arbitrary base point and $\tilde{p}_1$ respectively $\tilde{p}_2$ be paths in $\tilde{\Gamma}_{x_0,x_1,x_2}$ from V_0, the corresponding vertex to x_0, to V_1 respectively V_2. Since $\tilde{\Gamma}$, and

therefore $\tilde{\Gamma}_{x_0,x_1,x_2}$, is simply-connected, they differ exactly by $\tilde{e}_{1,2}$ and it can be calculated

$$\begin{aligned}\tilde{\iota}'(x_1) - \tilde{\iota}'(x_2) &= [., \pi'(\tilde{p}_1)] - [., \pi'(\tilde{p}_2)] \\ &= [., \pi'(\tilde{p}_1) - \pi'(\tilde{p}_2)] \\ &= [., \pi'(\tilde{p}_1 - \tilde{p}_2)] \\ &= [., \pi'(\pm\tilde{e}_{1,2})] \\ &= \pm[., \pi'(\tilde{e}_{1,2})] \\ &= \pm[., e_{1,2}] \\ &= 0\end{aligned}$$

as $[e_{1,2}, m] = 0$ for all $m \in H_1(\Gamma, \mathbb{Z})$ and $[., e_{1,2}] \in N_{\mathbb{R}} = \mathrm{Hom}(H_1(\Gamma, \mathbb{Z}), \mathbb{R})$. Since it holds $\tilde{\iota}'(x_1) = \tilde{\iota}'(x_2)$ for arbitrary points x_1, x_2 on $\tilde{e}$, $\tilde{\iota}'$ is constant on $\tilde{e}$.

(ii) First it has to be specified what is meant with affine linearity on $\tilde{e}$. $\tilde{\iota}'$ maps to $N_{\mathbb{R}}$, which is a $\mathbb{R}$-vector space. Thus one wants to have a kind of $\mathbb{R}$-linearity on $\tilde{e}$ to give a meaning to the term affine linearity for $\tilde{\iota}'$. Let x_0, x_1 be the vertices adjacent to $\tilde{e}$. From Construction 6.1.3 it is known already that the reduced edge $\pi'(\tilde{e})$, which is an edge of Γ, can be identified with the interval $(0, l(\pi'(\tilde{e})))$. As $\mathbb{R}$-linearity exists on $(0, l(\tilde{e}))$, it induces $\mathbb{R}$-linearity on $\pi'(\tilde{e})$ and hence on $\tilde{e}$, as they are isomorphic.
Also by Construction 6.1.3 follows: Choose x_0 in a way that it becomes identified with 0, and x_1 becomes identified with $l(\pi'(\tilde{e}))$ from the closed interval $[0, l(\pi'(\tilde{e}))]$. Let x be the point on $\tilde{e}$ identified with $t \cdot l(\pi'(\tilde{e}))$, where $t \in [0,1]$. Refinement creates an edge $\tilde{e}'$ with adjacent vertices x_0 and x. Its reduction $\pi'(\tilde{e}')$ possesses a length $l(\pi'(\tilde{e}'))$, too. But it holds $l(\pi'(\tilde{e}')) = t \cdot l(\pi'(\tilde{e}))$, which follows from Construction 6.1.3 by choosing $r_0 = t \cdot l(\pi'(\tilde{e}))$, which is identified with x.
As $\mathbb{R}$-linearity has a meaning for $\tilde{e}$, one can start to prove that $\tilde{\iota}'$ is affine-linear on $\tilde{e}$. Take x_0 as a base point. Then the path from x_0 to x is just the edge $\tilde{e}'$. By the identification of $\tilde{e}$ with $(0, l(\pi'(\tilde{e})))$, x can be seen as $t \cdot x_1$. Then it holds

$$\begin{aligned}\tilde{\iota}'(t \cdot x_1) &= \tilde{\iota}'(x) \\ &= \tilde{\iota}'(x) \\ &= [., \pi'(\tilde{e}')].\end{aligned}$$

A cycle $m \in H_1(\Gamma, \mathbb{Z})$ contains $m_e \in \mathbb{Z}$ times the edge $e = \pi'(\tilde{e})$. Hence $[m, \pi'(\tilde{e}')] = m_e l(\pi'(\tilde{e}'))$, which allows to continue the calculation

$$\begin{aligned}&= \frac{l(\pi'(\tilde{e}'))}{l(\pi'(\tilde{e}))} \cdot [., \pi'(\tilde{e})] \\ &= \frac{t \cdot l(\pi'(\tilde{e}))}{l(\pi'(\tilde{e}))} \cdot [., \pi'(\tilde{e})] \\ &= t \cdot [., \pi'(\tilde{e})] \\ &= t \cdot \tilde{\iota}'(x_1).\end{aligned}$$

Hence $\tilde{\iota}'$ is $\mathbb{R}$-linear on $\tilde{e}$ if the base point x_0 is chosen as above. If one chooses an arbitrary base point x_0', a constant has to be added that corresponds to the path from x_0' to x_0. Therefore $\tilde{\iota}'$ is just affine $\mathbb{R}$-linear on $\tilde{e}$.

□

Definition 6.1.17. Since $\tilde{\iota}'$ is affine $\mathbb{R}$-linear on $\tilde{e}$, there exists a $c \in N_{\mathbb{R}}$ and a linear map $\tilde{\iota}'_{\text{linear}} : \tilde{\Gamma} \longrightarrow N_{\mathbb{R}}$ such that $\tilde{\iota}'(x) = \tilde{\iota}'_{\text{linear}}(x) + c$. Furthermore $\tilde{\iota}'_{\text{linear}}$ is called the linear map corresponding to $\tilde{\iota}'$.

Theorem 6.1.18. *$\tilde{\iota}'_{\text{linear}}$ is bounded by the genus g of the curve X.*

Proof. Firstly, the genus g of the curve X is equal to the rank of the free abelian group $M = H_1(\Gamma, \mathbb{Z})$, where Γ is the corresponding graph to $\Gamma_{\mathcal{X}}$ and $\mathcal{X}$ is a semistable model for X.
Let $\tilde{e}$ be an arbitrary edge of $\tilde{\Gamma}$ and choose the base point $x_0' = x_0$, where the notation from the proof of Theorem 6.1.16 is used. Then $\tilde{\iota}'_{\text{linear}}$ and $\tilde{\iota}'$ coincide. As $\tilde{\iota}'_{\text{linear}} = \tilde{\iota}'$ is linear on $\tilde{e}$, it suffices to check the operator norm for the element x_1 (notation from the proof of Theorem 6.1.16).
The norm of x_1 is just the distance to x_0, namely $l(\pi'(\tilde{e})) \in \mathbb{R}_{>0}$. The norm of $\tilde{\iota}'_{\text{linear}}(x_1)$ depends on the chosen norm on $N_{\mathbb{R}} \cong \mathbb{R}^g$. If $\mathbb{R}^g$ is considered as an Euclidean space, one has the norm defined by the scalar product. In general, the sum norm is the biggest one on $\mathbb{R}^g$, so this norm will be considered to treat the worst case.

$$\begin{aligned} \alpha : N_{\mathbb{R}} &\longrightarrow \mathbb{R}^g, \\ \varphi &\longmapsto (\varphi(m_1), ..., \varphi(m_g)) \end{aligned}$$

delivers a canonical isomorphism between $N_{\mathbb{R}}$ and $\mathbb{R}^g$ after choosing a $\mathbb{Z}$-basis of M. By identifying them via this isomorphism, the image of x_1 with respect to $\tilde{\iota}'_{\text{linear}}$ is

$$\tilde{\iota}'_{\text{linear}}(x_1) = \begin{pmatrix} [m_1, \pi'(\tilde{e})] \\ ... \\ ... \\ [m_g, \pi'(\tilde{e})] \end{pmatrix}$$

where for each entry it holds

$$0 \leq [m_1, \pi'(\tilde{e})] \leq l(\pi'(\tilde{e})).$$

Hence

$$\left\| \tilde{\iota}'_{\text{linear}}(x_1) \right\|_1 = \left\| \begin{pmatrix} [m_1, \pi'(\tilde{e})] \\ ... \\ ... \\ [m_g, \pi'(\tilde{e})] \end{pmatrix} \right\|_1 \leq g \cdot l(\pi'(\tilde{e})) = g \cdot \|x_1\|$$

and

$$\left\| \tilde{\iota}'_{\text{linear}} \right\|_{\text{Op}} \leq g$$

by the remark in the beginning.

□

6.2 Tropicalizing the Abel-Jacobi map

The goal is now to investigate the relationship between the algebraic and tropical Abel-Jacobi map. Take the same presumptions as in section 6.1. The following results were proven in Theorem 4.3.7, [KRZ16a] under the additional assumption that X is a Mumford curve, meaning that $\mathcal{X}_k$ has only rational components. By Theorem 4.2.5, [KRZ16a] there exists then a homomorphism $\eta : H_1(X^{\mathrm{an}}, \mathbb{Z}) \longrightarrow T(\mathbb{C}_p)$ such that $J^{\mathrm{an}} \cong T^{\mathrm{an}}/\eta\,(H_1(X^{\mathrm{an}}, \mathbb{Z}))$. Contrary to this, in the following it will be done in greater generality by using the Raynaud uniformization theory, explained in section 2.4.

Construction 6.2.1. The aim is to construct the skeleton of the Jacobian J, belonging to the curve X. Since the Jacobian is an abelian variety, the following results will be proven in greater generality for any abelian variety A over a field K, described in the preamble of section 6.1.
By Construction 2.4.2 there exists the short exact sequence

$$0 \longrightarrow M' \longrightarrow E^{\mathrm{an}} \xrightarrow{\pi} A^{\mathrm{an}} \longrightarrow 0.$$

In Construction 2.4.4 the unique morphism

$$\mathrm{trop} : E^{\mathrm{an}} \longrightarrow N_{\mathbb{R}}$$

was defined. Since $M' = \ker(\pi)$ is a subset of E^{an}, one receives the following commutative diagram

$$\begin{tikzcd} M' \arrow[r] \arrow[d, "\mathrm{trop}"] & E^{\mathrm{an}} \arrow[d, "\mathrm{trop}"] \\ \mathrm{trop}(M') \arrow[r] & N_{\mathbb{R}} \end{tikzcd}$$

where $\mathrm{trop}(M')$ is just a subset of $N_{\mathbb{R}}$, and $\mathrm{trop}(M') \longrightarrow N_{\mathbb{R}}$ is consequently just the inclusion.
The tropicalization map yields now a unique map

$$\overline{\tau} : A^{\mathrm{an}} = E^{\mathrm{an}}/M' \longrightarrow \Sigma := N_{\mathbb{R}}/\,\mathrm{trop}(M')$$

such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r] \arrow[d, "\mathrm{trop}"] & E^{\mathrm{an}} \arrow[r, "\pi"] \arrow[d, "\mathrm{trop}"] & A^{\mathrm{an}} \arrow[r] \arrow[d, "\overline{\tau}"] & 0 \\ 0 \arrow[r] & \mathrm{trop}(M') \arrow[r] & N_{\mathbb{R}} \arrow[r] & \Sigma \arrow[r] & 0 \end{tikzcd}$$

commutes and both lines are exact.

$$\Sigma = N_{\mathbb{R}}/\,\mathrm{trop}(M')$$

is called the **skeleton of the Jacobian** if one sets $A = J$. In section 6.1, the Jacobian of the skeleton of X was already defined to be $\mathrm{Jac}(\Gamma) = N_{\mathbb{R}}/\eta'(H_1(\Gamma, \mathbb{Z}))$. Therefore the next goal is to show: The skeleton of the Jacobian is the Jacobian of the skeleton.

Lemma 6.2.2. *There exists a unique surjective homomorphism*

$$\overline{\tau}' : J^{\mathrm{an}} \longrightarrow \mathrm{Jac}(\Gamma)$$

such that the diagram

$$\begin{tikzcd} E^{\mathrm{an}} \arrow[r, "\pi"] \arrow[d, "\mathrm{trop}"] & J^{\mathrm{an}} \arrow[d, "\overline{\tau}'"] \\ N_{\mathbb{R}} \arrow[r, two heads] & \mathrm{Jac}(\Gamma) \end{tikzcd}$$

commutes.

Proof. Corollary 5.6, [BR14]. □

Theorem 6.2.3 (**The skeleton of the Jacobian is the Jacobian of the skeleton**). *There is a canonical isomorphism between the lattices*

$$\mathrm{trop}(M') \subseteq N_{\mathbb{R}}$$

and

$$\eta'\left(H_1(\Gamma, \mathbb{Z})\right) \subseteq N_{\mathbb{R}}.$$

Hence

$$\Sigma \cong \mathrm{Jac}(\Gamma).$$

Furthermore the maps $\overline{\tau} : J^{\mathrm{an}} \longrightarrow \Sigma$ *and* $\overline{\tau}' : J^{\mathrm{an}} \longrightarrow \mathrm{Jac}(\Gamma)$ *coincide under this identification.*

Proof. Corollary 6.6, [BR14]. □

Theorem 6.2.4 (**The retraction and Abel-Jacobi map commute**). *Let* $P_0 \in X(\mathbb{C}_p)$ *and let* $x_0 = \tau(P_0) \in \Gamma$. *Choose the algbebraic and tropical Abel-Jacobi maps with respect to these base points. Then the following square is commutative:*

$$\begin{tikzcd} X^{\mathrm{an}} \arrow[r, "\iota"] \arrow[d, "\tau"] & J^{\mathrm{an}} \arrow[d, "\overline{\tau}"] \\ \Gamma \arrow[r, "\iota'"] & \Sigma \end{tikzcd}$$

Proof. Follows from Proposition 6.1, [BR14] combined with the identification from Theorem 6.2.3. □

6.3 Comparing the integrals on a curve

The final goal is to investigate the difference between the abelian and Berkovich-Coleman integral on curves. Take the same presumptions as in section 6.1.

Construction 6.3.1. Let A be an abelian variety over $\mathbb{C}_p$ and let $\pi : E^{\mathrm{an}} \longrightarrow A^{\mathrm{an}}$ be the topological universal cover of A^{an}.
Since E^{an} is locally isomorphic to A^{an}, it holds $\mathrm{Lie}(A) = \mathrm{Lie}(E)$, and since any invariant one-form on E^{an} descends to an invariant one-form on A^{an}, one has $\Omega^1_{\mathrm{inv}}(E) = \Omega^1_{\mathrm{inv}}(A)$.
Normally the Berkovich-Coleman integral is path-dependent, but as E^{an} is simply-connected, any Berkovich-Coleman integral on E^{an} is path-independent and it is possible to define the homomorphism

$$\log_{\mathrm{BC}} : E(\mathbb{C}_p) \longrightarrow \mathrm{Lie}(A), P \longmapsto {}^{\mathrm{BC}}\!\!\int_0^P .$$

Restricting $\pi : E^{\mathrm{an}} \longrightarrow A^{\mathrm{an}}$ to $E(\mathbb{C}_p)$ and composing with $\log_{A(\mathbb{C}_p)} : A(\mathbb{C}_p) \longrightarrow \mathrm{Lie}(A)$ delivers the homomorphism

$$\log_{\mathrm{Ab}} : E(\mathbb{C}_p) \longrightarrow \mathrm{Lie}(A), P \longmapsto {}^{\mathrm{Ab}}\!\!\int_0^{\pi(P)} .$$

Theorem 6.3.2. *Let A be an abelian variety and $\pi : E^{\mathrm{an}} \longrightarrow A^{\mathrm{an}}$ be the lift of A^{an}. Then the homomorphism*

$$\begin{aligned} \log_{\mathrm{BC}} - \log_{\mathrm{Ab}} : E(\mathbb{C}_p) &\longrightarrow \mathrm{Lie}(A), \\ P &\longmapsto {}^{\mathrm{BC}}\!\!\int_0^P \pi^* - {}^{\mathrm{Ab}}\!\!\int_0^{\pi(P)} \end{aligned}$$

factorizes as

$$\begin{tikzcd} E(\mathbb{C}_p) \arrow[rr, "\log_{\mathrm{BC}} - \log_{\mathrm{Ab}}"] \arrow[dr, "\mathrm{trop}"'] & & \mathrm{Lie}(A) \\ & N_{\mathbb{Q}}. \arrow[ur, "L"'] & \end{tikzcd}$$

Proof. By Corollary 2.4.6 there exists the short exact sequence

$$0 \longrightarrow A_0(\mathbb{C}_p) \longrightarrow E(\mathbb{C}_p) \xrightarrow{\mathrm{trop}} N_{\mathbb{Q}} \longrightarrow 0.$$

The homomorphism $\log_{\mathrm{BC}} - \log_{\mathrm{Ab}}$ factorizes through $N_{\mathbb{Q}} = E(\mathbb{C}_p)/A_0(\mathbb{C}_p)$ if and only if $A_0(\mathbb{C}_p) \subseteq \ker(\log_{\mathrm{BC}} - \log_{\mathrm{Ab}})$, meaning that it remains to prove that

$$\log_{\mathrm{BC}} = \log_{\mathrm{Ab}}$$

on $A_0(\mathbb{C}_p)$:

$\log_{\mathrm{Ab}}$ is uniquely characterized by $\mathrm{d}\log_{\mathrm{Ab}} : \mathrm{Lie}(A_0) \longrightarrow \mathrm{Lie}(A_0)$ to be the identity, and it holds: For all open subgroups H in the canonical topology of $A_0(\mathbb{C}_p)$, $A_0(\mathbb{C}_p)/H$ is a torsion group. This is a consequence of the proof of Theorem 4.1.2.
A_0 is not an abelian variety anymore, so it is not possible to directly get a unique $\log_{\mathrm{Ab}}$. But by Lemma 4.1.15 there exists a unique logarithm

$$\log_{A_0} : A_0(\mathbb{C}_p) \longrightarrow \mathrm{Lie}(A_0) = \mathrm{Lie}(A)$$

with $\mathrm{d}\log_{A_0} = \mathrm{id}$ if $A_0(\mathbb{C}_p)$ is a $\mathbb{C}_p$-Lie group and $A_0(\mathbb{C}_p)_f = A_0(\mathbb{C}_p)$ (notation of Definition 4.1.12). The first follows because $A_0(\mathbb{C}_p)$ is a $\mathbb{C}_p$-analytic domain in $A(\mathbb{C}_p)$, and hence an open subgroup of $A(\mathbb{C}_p)$ in the naive analytic topology. The second holds because $A_0(\mathbb{C}_p)/H$ is a torsion group for any open subgroup H. Therefore $\log_{\mathrm{Ab}}|_{A_0(\mathbb{C}_p)} = \log_{A_0}$ is uniquely characterized by the fact that its linearization is the identity map.
Consider $\log_{\mathrm{BC}}$. If it were shown that $\log_{\mathrm{BC}}$ also induces the identity map on tangent spaces, the above equality would have been proven. Since

$$A_0(\mathbb{C}_p) = \ker(\mathrm{trop} : E(\mathbb{C}_p) \twoheadrightarrow N_{\mathbb{Q}}),$$

the deformation retraction of A^{an} takes A_0 onto $\{0\}$. Hence A_0 is simply-connected, as it is contractible. It follows that the Berkovich-Coleman integral is path-independent on $A_0(\mathbb{C}_p)$. With that said, $\log_{\mathrm{BC}}$ is well-defined on $A_0(\mathbb{C}_p)$.
A priori $\log_{\mathrm{BC}}$ could possibly be another logarithm than $\log_{A_0}$. The abelian logarithm $\log_{A_0}$ is uniquely defined by the property $\mathrm{d}\log_{A_0} = \mathrm{id}$. Hence it remains to show $\mathrm{d}\log_{\mathrm{BC}} = \mathrm{id}$. The linearization of $\log_{\mathrm{BC}}$ is defined on the Lie algebra of A_0, that is the tangent space at 0. The element $0 \in A_0 \subseteq A$ has a neighbourhood U, isomorphic to an open unit ball. On U the Berkovich-Coleman integral can be calculated by formal antidifferentation. Theorem 5.1.11 yields that the linearization of $\log_{\mathrm{BC}}$ is the identity because 1 is the neutral element of $\mathrm{Lie}(A_0)$. So it must be equal to $\log_{\mathrm{Ab}}$ on U as $\mathrm{d}\log_{\mathrm{BC}} : \mathrm{Lie}(A_0) \longrightarrow \mathrm{Lie}(A_0)$ is the identity.
Since $\log_{\mathrm{BC}}$ also fulfils the condition $\mathrm{d}\log_{\mathrm{BC}} = \mathrm{id}$, which is necessary for the uniqueness of the abelian logarithm, it follows $\log_{\mathrm{BC}} = \log_{\mathrm{Ab}}$ on $A_0(\mathbb{C}_p)$. ☐

Corollary 6.3.3. *The Berkovich-Coleman and abelian integral coincide on abelian varieties of good reduction.*

Proof. Good reduction means that there exists a model $\mathcal{A}$ of the abelian variety A with smooth special fibre $\mathcal{A}_k$, meaning that singular points do not exist. This is equivalent to the fact that the analytification A^{an} of the corresponding generic fibre $A = \mathcal{A}_K$ does not contain open annuli which are not subset of an open ball, meaning that its open annuli are all subsets of open balls and hence there are no loops in A^{an}. Since A is connected, it is contractible and hence $M = H_1(A^{\mathrm{an}}, \mathbb{Z}) = 0$. The same holds for its dual $N = \mathrm{Hom}(M, \mathbb{Z}) = 0$. Eventually $\log_{\mathrm{BC}} - \log_{\mathrm{Ab}} : E(\mathbb{C}_p) \longrightarrow \mathrm{Lie}(A)$ factorizes through $\{0\}$, meaning that $\log_{\mathrm{BC}} - \log_{\mathrm{Ab}} = 0$ on the whole set $E(\mathbb{C}_p)$. ☐

Corollary 6.3.4. *The Berkovich-Coleman and abelian integral coincide on open balls contained in X^{an}.*

Proof. An open ball B in the analytification X^{an} always retracts to a single point of the skeleton, consequently we have

$$\mathrm{trop}(B) = \xi \in N_{\mathbb{Q}},$$

which leads to

$$\begin{aligned}(\log_{\mathrm{BC}} - \log_{\mathrm{Ab}})(B) &= (L \circ \mathrm{trop})(B) \\ &= L(\xi) \in \mathrm{Lie}(A).\end{aligned}$$

This means that $\log_{\mathrm{BC}}$ and $\log_{\mathrm{Ab}}$ on U just differ by a constant, meaning that there exists a $c \in \mathrm{Lie}(A)$ such that $\log_{\mathrm{BC}} = \log_{\mathrm{Ab}} + c$. For two points $P, Q \in U$ it holds

$$\begin{aligned}{}^{\mathrm{BC}}\!\int_P^Q \omega &= {}^{\mathrm{BC}}\!\int_0^Q \omega \;-\; {}^{\mathrm{BC}}\!\int_0^P \omega \\ &= \log_{\mathrm{BC}}(Q)\ \omega - \log_{\mathrm{BC}}(P)\ \omega \\ &= (\log_{\mathrm{Ab}}(Q) + c)\ \omega - (\log_{\mathrm{Ab}}(P) + c)\ \omega \\ &= \log_{\mathrm{Ab}}(Q)\ \omega - \log_{\mathrm{Ab}}(P)\ \omega \\ &= {}^{\mathrm{Ab}}\!\int_0^{\pi(Q)} \omega \;-\; {}^{\mathrm{Ab}}\!\int_0^{\pi(P)} \omega \\ &= {}^{\mathrm{Ab}}\!\int_{\pi(P)}^{\pi(Q)} \omega\end{aligned}$$

for any $\omega \in \Omega^1_{\mathrm{inv}}(E)$.
Since U is already simply-connected and the path was chosen in U, the equation can be reduced from E^{an} to A^{an} and it follows

$${}^{\mathrm{BC}}\!\int_P^Q \omega = {}^{\mathrm{Ab}}\!\int_P^Q \omega$$

for any $\omega \in \Omega^1_{\mathrm{inv}}(A)$. □

Construction 6.3.5. The goal is to also control the difference of those integrals between two $\mathbb{C}_p$-points not contained in the same open ball. Unfortunately one cannot control this difference between two arbitrary $\mathbb{C}_p$-points, but it is possible to do so within an open annulus of the semistable decomposition.
For this purpose the commutative diagram from Theorem 6.2.4 will be lifted:

$$\begin{array}{ccc} \tilde{X}^{\mathrm{an}} & \xrightarrow{\tilde{\iota}} & E^{\mathrm{an}} \\ \big\downarrow{\scriptstyle \tau} & & \big\downarrow{\scriptstyle \mathrm{trop}} \\ \tilde{\Gamma} & \xrightarrow{\tilde{\iota}'} & N_{\mathbb{R}} \end{array}$$

Let $\tilde{e} \subseteq \tilde{\Gamma}$ be an open edge, with image $e \subseteq \Gamma$. Define

$$A := \tau^{-1}(e)$$

and

$$\tilde{A} := \tau^{-1}(\tilde{e}).$$

These are open annuli and

$$A \cong \tilde{A}.$$

This yields the following result:

Theorem 6.3.6. *With the above notation, there is a $\mathbb{C}_p$-linear map*

$$a : \Omega^1_{X/\mathbb{C}_p} \longrightarrow \mathbb{C}_p$$

such that, for all $P, Q \in A(\mathbb{C}_p)$, it holds

$$^{\mathrm{BC}}\!\!\int_P^Q \omega - {}^{\mathrm{Ab}}\!\!\int_P^Q \omega = a(\omega)\left(v(Q) - v(P)\right).$$

Proof. As P and Q both lie in an open annulus that is simply-connected, the Berkovich-Coleman integration is path-independent and $^{\mathrm{BC}}\!\int_P^Q \omega$ makes sense. Choose P_0 and $x_0 = \tau(P_0)$ as the base points of the Abel-Jacobi maps. Furthermore $\tilde{P_0}$ is chosen to be the lift of P_0, such that $\tilde{\iota}(\tilde{P_0}) = 0$. This is possible as $\iota(P_0) = 0$. With $\omega_J \in \Omega^1_{J/\mathbb{C}_p}$ and Proposition 6.3.2, applied to the Jacobian J, it follows

$$\begin{aligned}
&{}^{\mathrm{BC}}\!\!\int_{\tilde{\iota}(\tilde{P_0})}^{\tilde{\iota}(\tilde{Q})} \pi^*\omega_J - {}^{\mathrm{Ab}}\!\!\int_{\tilde{\iota}(\tilde{P_0})}^{\tilde{\iota}(\tilde{Q})} \pi^*\omega_J \\
&= {}^{\mathrm{BC}}\!\!\int_{\tilde{\iota}(\tilde{P_0})}^{\tilde{\iota}(\tilde{Q})} \pi^*\omega_J - {}^{\mathrm{Ab}}\!\!\int_{\pi(\tilde{\iota}(\tilde{P_0}))}^{\pi(\tilde{\iota}(\tilde{Q}))} \omega_J \\
&= {}^{\mathrm{BC}}\!\!\int_0^{\tilde{\iota}(\tilde{Q})} \pi^*\omega_J - {}^{\mathrm{Ab}}\!\!\int_0^{\pi(\tilde{\iota}(\tilde{Q}))} \omega_J \\
&\quad - \left({}^{\mathrm{BC}}\!\!\int_0^{\tilde{\iota}(\tilde{P_0})} \pi^*\omega_J - {}^{\mathrm{Ab}}\!\!\int_0^{\pi(\tilde{\iota}(\tilde{P_0}))} \omega_J \right) \\
&= \left(\log_{\mathrm{BC}} - \log_{\mathrm{Ab}}\right)(\tilde{\iota}(\tilde{Q}))(\omega_J) \\
&\quad - \left(\log_{\mathrm{BC}} - \log_{\mathrm{Ab}}\right)(\tilde{\iota}(\tilde{P_0}))(\omega_J) \\
&= \left(\log_{\mathrm{BC}} - \log_{\mathrm{Ab}}\right)(\tilde{\iota}(\tilde{Q}) - \tilde{\iota}(\tilde{P_0}))(\omega_J) \\
&= \left(\log_{\mathrm{BC}} - \log_{\mathrm{Ab}}\right)(\tilde{\iota}(\tilde{Q}))(\omega_J) \\
&\overset{6.3.2}{=} \left(L \circ \mathrm{trop}\right)(\tilde{\iota}(\tilde{Q}))(\omega_J) \\
&= \left(L \circ \mathrm{trop} \circ \tilde{\iota}\right)(\tilde{Q})(\omega_J) \\
&\overset{6.3.5}{=} \left(L \circ \tilde{\iota}' \circ \tau\right)(\tilde{Q})(\omega_J) = (*)
\end{aligned}$$

For a $\mathbb{C}_p$-point on an open annulus, the retraction map is just the valuation map $P \longmapsto v(P)$ if we choose an isomorphism

$$\zeta_{\mathrm{go}} : A \xrightarrow{\sim} S(\rho)_+.$$

The image $v(P)$ can again be identified with a point of the skeleton. The situation can be described with a commutative diagram, where ζ_{go} and ζ_{return} are isomorphisms.

$$\begin{tikzcd}
\tilde{A} \subseteq \tilde{X}^{\mathrm{an}} \arrow[r, "\tau"] \arrow[d, "\zeta_{\mathrm{go}}"] & \tilde{\Gamma} \\
S(\rho)_+ \arrow[r, "v"] & (v(1), v(\rho)) \arrow[u, "\zeta_{\mathrm{return}}"']
\end{tikzcd}$$

$$\begin{aligned}
(*) &= \left(L \circ \tilde{\iota}' \circ \zeta_{\mathrm{return}} \circ v \circ \zeta_{\mathrm{go}}\right)(\tilde{Q})(\omega_J) \\
&= \left(L \circ \tilde{\iota}'\right)\left(\zeta_{\mathrm{return}} \circ v \circ \zeta_{\mathrm{go}}(\tilde{Q})\right)(\omega_J)
\end{aligned}$$

$\tilde{\iota}'$ is affine-linear on $\tilde{e}$, hence there exist $c \in N_{\mathbb{R}}$ and a linear map $\tilde{\iota}'_{\mathrm{linear}} : \tilde{\Gamma} \longrightarrow N_{\mathbb{R}}$, such that $\tilde{\iota}'(x) = \tilde{\iota}'_{\mathrm{linear}}(x) + c$.

$$= L\left(\tilde{\iota}'_{\mathrm{linear}}\left(\zeta_{\mathrm{return}} \circ v \circ \zeta_{\mathrm{go}}(\tilde{Q})\right) + c\right)(\omega_J)$$

It exists an isomorphism $\iota^* : \Omega^1_{J/\mathbb{C}_p} \longrightarrow \Omega^1_{X/\mathbb{C}_p}$. Hence, for $\omega \in \Omega^1_{X/\mathbb{C}_p}$ follows:

$$\begin{aligned}
& {}^{\mathrm{BC}}\!\!\int_P^Q \omega - {}^{\mathrm{Ab}}\!\!\int_P^Q \omega \\
&= {}^{\mathrm{BC}}\!\!\int_{\tilde{\iota}(\tilde{P})}^{\tilde{\iota}(\tilde{Q})} \pi^*\iota^*\omega - {}^{\mathrm{Ab}}\!\!\int_{\tilde{\iota}(\tilde{P})}^{\tilde{\iota}(\tilde{Q})} \pi^*\iota^*\omega \\
&= {}^{\mathrm{BC}}\!\!\int_{\tilde{\iota}(\tilde{P}_0)}^{\tilde{\iota}(\tilde{Q})} \pi^*\iota^*\omega - {}^{\mathrm{Ab}}\!\!\int_{\tilde{\iota}(\tilde{P}_0)}^{\tilde{\iota}(\tilde{Q})} \pi^*\iota^*\omega \\
&\quad - \left({}^{\mathrm{BC}}\!\!\int_{\tilde{\iota}(\tilde{P}_0)}^{\tilde{\iota}(\tilde{P})} \pi^*\iota^*\omega - {}^{\mathrm{Ab}}\!\!\int_{\tilde{\iota}(\tilde{P}_0)}^{\tilde{\iota}(\tilde{P})} \pi^*\iota^*\omega \right) \\
&= L\left(\tilde{\iota}'_{\mathrm{linear}}\left(\zeta_{\mathrm{return}} \circ v \circ \zeta_{\mathrm{go}}(\tilde{Q})\right) + c\right)(\iota^*\omega) \\
&\quad - L\left(\tilde{\iota}'_{\mathrm{linear}}\left(\zeta_{\mathrm{return}} \circ v \circ \zeta_{\mathrm{go}}(\tilde{P})\right) + c\right)(\iota^*\omega) \\
&= L\left(\tilde{\iota}'_{\mathrm{linear}}\left(\zeta_{\mathrm{return}} \circ v \circ \zeta_{\mathrm{go}}(\tilde{Q} - \tilde{P})\right)\right)(\iota^*\omega)
\end{aligned}$$

Under the isomorphism ζ_{go}, the open annulus $\tilde{A}$ will be identified with $S(\rho)_+$. Therefore $P = \zeta_{\mathrm{go}}(\tilde{P})$ and $Q = \zeta_{\mathrm{go}}(\tilde{Q})$.

$$= \left(L \circ \tilde{\iota}'_{\mathrm{linear}} \circ \zeta_{\mathrm{return}}\right)(v(Q) - v(P))(\iota^*\omega)$$

As it was already identified $\tilde{A}$ with $S(\rho)_+$, it makes sense to identify $(v(1), v(\rho))$ with $\zeta_{\text{return}}((v(1), v(\rho))) \subseteq \tilde{\Gamma}$, too. This permits quitting ζ_{return}.

$$= \left(L \circ \tilde{\iota}'_{\text{linear}}\right)(v(Q) - v(P))(\iota^* \omega)$$

Since L and $\tilde{\iota}'_{\text{linear}}$ are both $\mathbb{Q}$-linear, it follows

$$= (v(Q) - v(P)) \cdot \left(L \circ \tilde{\iota}'_{\text{linear}}\right)(1)(\iota^* \omega)$$

with $\left(L \circ \tilde{\iota}'_{\text{linear}}\right)(1) \in \text{Lie}(J) = (\Omega^1_{J/\mathbb{C}_p})^*$. Note that $\tilde{\iota}'_{\text{linear}}(1)$ is well-defined because 1 corresponds to a vertex adjacent to the retraction of $\tilde{A}$, where $\tilde{\iota}'_{\text{linear}}$ is defined, too. Furthermore L is defined at $\tilde{\iota}'_{\text{linear}}(v(Q) - v(P))$, since $v(Q) - v(P) \in \mathbb{Q}$ and hence $\tilde{\iota}'_{\text{linear}}(v(Q) - v(P)) \in N_{\mathbb{Q}}$. As ι^* is an isomorphism,

$$a := \left(\left(L \circ \tilde{\iota}'_{\text{linear}}\right)(1)\right) \circ \iota^*$$

is a $\mathbb{C}_p$-linear map $\Omega^1_{X/\mathbb{C}_p} \longrightarrow \mathbb{C}_p$. Continuing the calculation with these results gives:

$$= (v(Q) - v(P)) \cdot a(\omega)$$

□

Corollary 6.3.7. *Let V be the subspace of $\Omega^1_{X/\mathbb{C}_p}$ consisting of all ω such that*

$$^{\text{BC}}\!\!\int_P^Q \omega = {}^{\text{Ab}}\!\!\int_P^Q \omega$$

for all $P, Q \in A(\mathbb{C}_p)$, where A is an open annulus. Then V has codimension at most one.

Proof. It holds $V = \ker(a)$, where a is the $\mathbb{C}_p$-linear map $a : \Omega^1_{X/\mathbb{C}_p} \longrightarrow \mathbb{C}_p$ of Theorem 6.3.6. One may consider $\Omega^1_{X/\mathbb{C}_p}$ and $\mathbb{C}_p$ as $\mathbb{C}_p$-vector spaces.
Then, by the rank-nullity theorem, it holds

$$\dim_{\mathbb{C}_p}(\Omega^1_{X/\mathbb{C}_p}) = \dim_{\mathbb{C}_p}(\ker(a)) + \dim_{\mathbb{C}_p}(\text{Im}(a)).$$

Hence

$$\begin{aligned} \dim_{\mathbb{C}_p}(V) &= \dim_{\mathbb{C}_p}(\ker(a)) \\ &= \dim_{\mathbb{C}_p}(\Omega^1_{X/\mathbb{C}_p}) - \dim_{\mathbb{C}_p}(\text{Im}(a)) \end{aligned}$$

and

$$\text{codim}_{\mathbb{C}_p}(V) = \dim_{\mathbb{C}_p}(\text{Im}(a)) \leq 1$$

as $\text{Im}(a) \subseteq \mathbb{C}_p$. □

Remark 6.3.8. This result is an important ingredient in the proof of a theorem of Michael Stoll (Theorem 9.1, [Sto13]), which uniformely bounds the number of $\mathbb{Q}$-rational points on a smooth hyperelliptic curve of genus g with Mordell-Weil rank $\mathrm{rank}_{\mathbb{Z}}(J(\mathbb{Q})) \leq g-3$.
Furthermore it is a very important tool in the proof of Theorem 2.14, [KRZ16a]. There Eric Katz, Joseph Rabinoff and David Zureick-Brown could overcome the hyperelliptic restriction in Theorem 9.1, [Sto13] and formulate uniform bounds of the number of $\mathbb{Q}$-rational points for arbitrary smooth curves X of genus g with Mordell-Weil rank $\mathrm{rank}_{\mathbb{Z}}(J(\mathbb{Q})) \leq g-3$. The statement in Theorem 2.14, [KRZ16a] is

$$\#X(\mathbb{Q}) \leq 84g^2 - 98g + 28.$$

References

[BF10] M. Baker and X. Faber. "Metric properties of the tropical Abel-Jacobi map". In: *Journal of Algebraic Combinatorics* 33.3 (July 2010), pp. 349–381.

[BPR13] M. Baker, S. Payne, and J. Rabinoff. "On the structure of non-Archimedean analytic curves". In: *Tropical and non-Archimedean geometry*. Vol. 605. Contemp. Math. Amer. Math. Soc., Providence, RI, 2013, pp. 93–121.

[BR14] M. Baker and J. Rabinoff. "The Skeleton of the Jacobian, the Jacobian of the Skeleton, and Lifting Meromorphic Functions From Tropical to Algebraic Curves: Fig. 1." In: *International Mathematics Research Notices* 2015.16 (Oct. 2014), pp. 7436–7472.

[BBK10] J. S. Balakrishnan, R. W. Bradshaw, and K. S. Kedlaya. "Explicit Coleman Integration for Hyperelliptic Curves". In: *Lecture Notes in Computer Science*. Springer Berlin Heidelberg, 2010, pp. 16–31.

[Ber90] V. G. Berkovich. *Spectral Theory and Analytic Geometry over Non-Archimedean Fields (Mathematical Surveys and Monographs)*. American Mathematical Society, 1990. ISBN: 0-8218-1534-2.

[Ber07] V. G. Berkovich. *Integration of One-forms on P-adic Analytic Spaces. (AM-162)*. Princeton University Press, Jan. 2007.

[Ber09] V. G. Berkovich. *Non-archimedean analytic spaces*. Short account of lectures at the Advanced School on p-adic Analysis and Applications. 2009. URL: `www.wisdom.weizmann.ac.il/~vova/Trieste_2009.pdf` (visited on Oct. 18, 2017).

[BL91] S. Bosch and W. Lütkebohmert. "Degenerating abelian varieties". In: *Topology* 30.4 (1991), pp. 653–698.

[Bou98] N. Bourbaki. *Lie Groups and Lie Algebras: Chapters 1-3*. Springer-Verlag GmbH, Aug. 3, 1998. ISBN: 3540642420.

[Col82] R. F. Coleman. "Dilogarithms, regulators and p-adic L-functions". In: *Inventiones Mathematicae* 69.2 (June 1982), pp. 171–208.

[GM] G. van der Geer and B. Moonen. *Abelian Varieties*. Draft for a book. URL: `https://www.math.ru.nl/~bmoonen/research.html` (visited on Oct. 18, 2017).

[GR02] A. Grothendieck and M. Raynaud. *Revêtements étales et groupe fondamental (SGA 1)*. June 20, 2002. arXiv: `math/0206203v2 [math.AG]`.

[Gub10] W. Gubler. "Non-Archimedean canonical measures on abelian varieties". In: *Compositio Mathematica* 146.3 (2010), pp. 683–730. ISSN: 0010-437X.

[Har77] R. Hartshorne. *Algebraic geometry*. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977, pp. xvi+496. ISBN: 0-387-90244-9.

[Haz78] M. Hazewinkel. *Formal groups and applications*. Vol. 78. Pure and Applied Mathematics. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1978, xxii+573pp. ISBN: 0-12-335150-2.

[KRZ16a] E. Katz, J. Rabinoff, and D. Zureick-Brown. *Diophantine and tropical geometry, and uniformity of rational points on curves*. June 30, 2016. arXiv: `1606.09618v2 [math.NT]`.

[KRZ16b] E. Katz, J. Rabinoff, and D. Zureick-Brown. "Uniform bounds for the number of rational points on curves of small Mordell-Weil rank". In: *Duke Mathematical Journal* 165.16 (2016), pp. 3189–3240. ISSN: 0012-7094.

[Sch11] P. Schneider. *p-adic Lie groups*. Vol. 344. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer, Heidelberg, 2011, pp. xii+254. ISBN: 978-3-642-21146-1.

[Sto13] M. Stoll. *Uniform bounds for the number of rational points on hyperelliptic curves of small Mordell-Weil rank*. July 6, 2013. arXiv: `1307.1773v6 [math.NT]`.

[Wer] A. Werner. *Skript zur Vorlesung Nicht-archimedische Geometrie*. Wintersemester 2014/2015. URL: `https://www.uni-frankfurt.de/52355122/nicht-archi-geom1415.pdf` (visited on Oct. 18, 2017).

[Zar96] Y. G. Zarhin. "p-adic abelian integrals and commutative Lie groups". In: *Journal of Mathematical Sciences* 81.3 (1996). Algebraic geometry, 4, pp. 2744–2750. ISSN: 1072-3374.

Selbstständigkeitserklärung

Ich habe die Arbeit selbstständig verfasst, keine anderen als die angegebenen Quellen und Hilfsmittel benutzt und bisher keiner anderen Prüfungsbehörde vorgelegt.
Die Grafiken wurden von mir selbst mit Hilfe des Programms *Inkscape* erstellt.

Regensburg, den 19.10.2017

Stefan Stadlöder